Characterization of the Class of Optimal Dense Circulant Graphs of Degree Four

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Abstract— **The class of dense circulant graphs of degree four with optimal distance-related properties is analysed in this paper. An algebraic study of this class is done. Two geometric characterizations are given, one in the plane and other in the space. Both characterizations facilitate the analysis of their topological properties and corroborate their suitability for implementing interconnection networks for distributed and parallel computers. Also a distance-hereditary non-disjoint decomposition of these graphs into rings is computed. Besides its practical consequences, this decomposition allows us the presentation of these optimal circulant graphs as a particular evolution of the traditional ring topology**

Index Terms— **Interconnection Networks, Parallel and Distributed Systems, Rings, Circulant Graphs.**

I. INTRODUCTION

RING topologies have frequently been used to implement local and campus area networks as well as other interconnection subsystems for diverse digital devices. In order to improve their relatively poor performance and survivability, extensions of rings have been considered in the literature such as Chordal Rings of degree three and four [1],[5]. The Torus is another low-degree regular network, based on a collection of rings, often used to interconnect highly coupled parallel systems. Some of the latest parallel computers using this topology are the IBM BlueGene/L supercomputer [2] and different multiprocessor servers based on the Alpha 21364 microprocessor [14].

Interconnection networks can be modelled by graphs with vertices representing their processing elements and edges representing the communication links between them. The performance and robustness of any network are mostly determined by the topological characteristics of its underlying graph. Thus, the selection of the most appropriate topology for a distributed or a parallel system is a critical design issue.

Circulant graphs have deserved significant attention during the last decades. The traditional ring and the complete graph topologies belong to this class of graphs. From a practical point of view, circulant graphs have been employed for a number of assorted applications. Circulants of different degrees constituted the basis of some classical distributed and parallel systems [16], [7]. The design of certain data alignment networks for complex memory systems have also relied on circulant graphs [17].

A class of circulant graphs of degree four with minimal topological distances, denoted as *Midimew* networks, was presented in [4] as a basis for building optimal interconnection networks for parallel computers. One *Midimew* network exists for any given number of nodes, *N*, which is completely defined by means of a single parameter, as described later. These graphs are *optimal* because they have the minimum average distance among all circulant graphs of degree four; consequently, the diameter of a *Midimew* graph is also minimum. In addition, these graphs are regular, vertex-symmetric and maximally connected, which means that they possess a very high degree of fault tolerance. Optimal VLSI layouts for these networks have also been explored [9] and some instances of *Midimew* graphs have recently been used as a basis for designing networks for massively parallel computers [18]. It is known that some instances of *Midimew* graphs are isomorphic to Chordal Rings of degree four, thus sharing their topological distance properties; however, there exist infinite values of N for which Midimew networks have smaller diameter and smaller average distance [5].

In this paper, we focus exclusively on dense circulant graphs of degree four. For every integer k, there exists a dense optimal circulant graph composed of $2k^2 + 2k + 1$ nodes, being k the diameter of the graph. We prove that any optimal dense circulant graph of degree four with diameter k is isomorphic to a Chordal Ring whose chord has length $2k+1$. In particular, the corresponding *Midimew* graph is also isomorphic to this Chordal Ring.

After proving the isomorphism, we will focus on two geometric characterizations of these optimal graphs that facilitate the analysis of their properties, especially those related to topological distances. It is well known that multiple parallel applications rely on the study of graph embedding techniques as a fundamental prerequisite to obtaining fast computational solutions. As a by-product of the above-mentioned geometric characterizations, we present in this paper a distance-hereditary decomposition of dense optimal Chordal Rings into a set of traditional rings. All the rings belonging to this set have the same number of nodes and their diameter corresponds to the diameter of the Chordal Ring in which they are embedded. The members of this embedded set of rings are non-disjoint and preserve the minimal routing of the original circulant graph. Besides its practical consequences in decomposing parallel algorithms and in the management of a parallel computer in space/time sharing, our research allows the

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presentation of these optimal graphs as a particular evolution of the traditional ring topology.

The rest of the paper is organized as follows: Section 2 is devoted to introducing some previous notation and for proving that all the dense optimal circulant graphs of degree four are, actually, the same graph. Section 3 explores some geometric characterizations, in both 2 and 3 dimensions, which provide a better understanding of the nature of these graphs and enlarge their application scope. Section 4 is dedicated to establishing a new distance-hereditary decomposition of such optimal Chordal Rings. The main conclusions obtained from this research are summarized in Section 5.

II. DENSE OPTIMAL CHORDAL RINGS OF DEGREE FOUR

A *circulant graph* with N vertices and jumps $\{j_1, j_2, \ldots, j_m\}$ is an undirected graph in which each vertex n, $0 \le n \le N-1$, is adjacent to all the vertices $n \pm j_i \bmod N$, with $1 \le i \le m$. We denote this graph as $C_N(j_1, j_2, \ldots, j_m)$. The family of circulant graphs includes the complete graph and the cyclic graph (ring) among its members.

We say that a circulant is *dense* if it has the maximum possible number of nodes for a given diameter. Thus, if k is a positive integer, a dense ring or $C_N(1)$ of degree two has $2k + 1$ nodes. There is another non-dense ring with $2k$ nodes. When adding another jump to the list, every integer k defines a family of $4k$ optimal (with minimum average distance and therefore minimum diameter) $C_N(j_1, j_2)$ graphs or *Midimew* networks of diameter k, with $j_1 = b-1$ and $j_2 = b$, where $b = \left\lceil \sqrt{\frac{N}{2}} \right\rceil$, see [4]. The dense member of this family contains $2k^2 + 2k + 1$ nodes. The other $4k$-1 graphs in the family correspond to non-dense values of N.

Two graphs G and H are isomorphic ($G \cong H$) if there exists a bijective mapping between their node sets which preserves adjacency. Is easy to prove the following:

Lemma — *Let k be a positive integer,* $N = 2k^2 + 2k + 1$. *If* $C_N(c,d)$ *is an optimal circulant graph, then* $C_N(c,d) \cong C_N(k, k+1)$.

Now, we consider the Chordal Ring $C_N(1, 2k+1)$. This Chordal Ring is isomorphic to the dense *Midimew* graph and hence, it is also optimal. To prove this claim we are going to use the following well-known theorem:

Theorem — [1] Let N be a natural number. We have $C_N(j_1, j_2) \cong C_N(i_1, i_2)$ if and only if there exists an integer u that $\gcd(u, N) = 1$ and

$$u\{\pm j_1, \pm j_2\} = \{\pm i_1, \pm i_2\} \bmod N.$$

Hence, the element $u = (k+1)^{-1} = -2k$ provides the adequate isomorphism between the two graphs.

For the rest of the paper we are going to consider $C_N(1, 2k+1)$ Chordal Rings as the representative members of the class of dense and optimal circulant graphs of degree four. A dense optimal Chordal Ring can be seen in Figure 1.

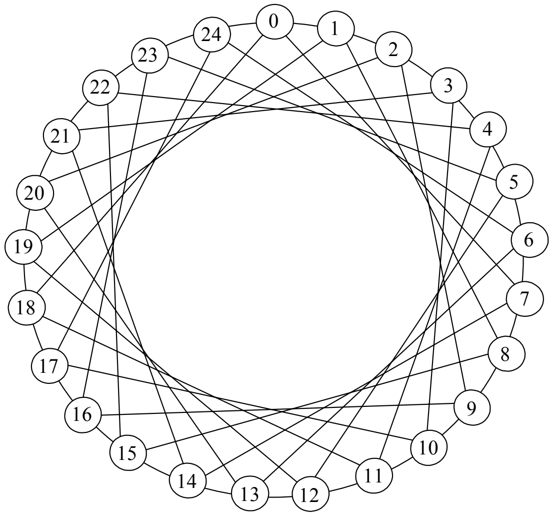


Figure 1: Dense Optimal Chordal Ring .

III. GEOMETRICAL CHARACTERIZATION OF OPTIMAL CHORDAL RINGS

In this section we will present two geometrical characterizations of $C_N(1, 2k+1)$ optimal graphs, in two and three dimensions, that lead to two physical implementations of interconnection networks based on these graphs.

The bi-dimensional characterization provides a better understanding of the topological nature of these graphs. The symmetry and the distance-related properties of these graphs are undoubtedly exposed when using this bi-dimensional geometric approach. We can use this characterization, among other applications, to provide an intuitive view of the network diameter, a sketch of a broadcasting algorithm, and a way to systematically vary the planar geometry of these graphs. In the next Section we provide a distance-hereditary decomposition into rings of $C_N(1, 2k+1)$ graphs as yet another by-product of our bi-dimensional geometric approach.

The three-dimensional characterization of $C_N(1, 2k+1)$ optimal graphs, suggests futuristic implementations of these optimal graphs over a torus' surface, whose nodes can optically communicate through free space. Our three-dimensional approach allows a new network implementation in which the physical distances are minimum as well and coincide with the topological ones. By using this 3-D characterization we are able to design minimum distance interconnection networks of degree four in which all the distances between any adjacent two nodes are always the same. In addition, such topological distances correspond to the Euclidean distance between any pair of adjacent nodes.

Let us begin with the bi-dimensional geometric characterization. By orthogonally arranging most of the $2N$ graph links, as in a mesh, we can obtain a particular representation of $C_{25}(1,7)$, the densest achievable graph $C_N(j_1, j_2)$ for $k = 3$, as shown in Figure 2. This layout requires $4k+2$ wrap-around links in order to completely interconnect the peripheral nodes.

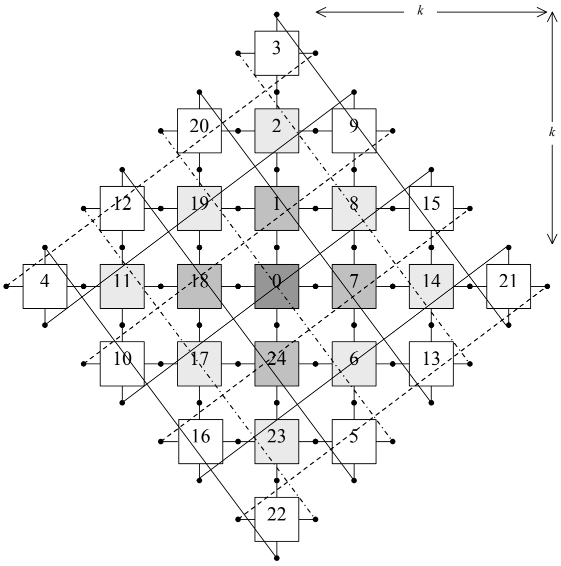


Figure 2: A Mesh-like Lay-out of a $C_{25}(1,7)$ Graph.

The vertex-symmetry of graphs allows their analysis from any vertex; node 0 will be used in the rest of the paper. In these graphs, there are $4d$ different nodes at distance d from node 0, with $1 \le d \le k$, and consequently, the total number of nodes of the graph is:

$$N = 1 + 4\sum_{d=1}^{k} d = 1 + 4\left(\frac{k(k+1)}{2}\right) = k^2 + (k+1)^2$$

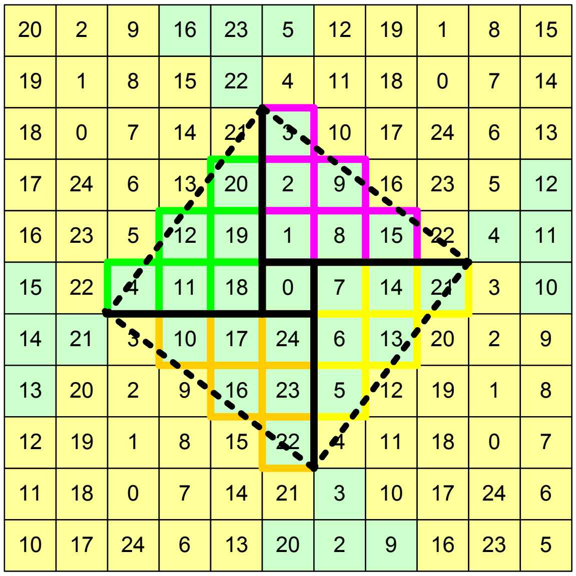
It is known that certain families of graphs can be fully represented by plane tessellations when their nodes are associated with regular polygons [7]. If each node in a $C_N(1,2k+1)$ graph is associated with a unitary square with its four neighbours attached to its sides as in [16], the graph can be characterized by a serrated or discrete square tile of area N, like the one in Figure 3. This tile tessellates the plane, and the network wrap-around links are defined according to the periodical pattern dictated by such a tessellation.

The last equation shows N to be the sum of two consecutive squares. A popular geometric proof of the Pythagorean theorem, which is clearly reflected in the left side of this equation, shows the square of the hypotenuse of a right-angled triangle of legs k and $k + 1$ as composed of four copies of such a right-angled triangle plus a central square with side $k+1-k=1$. The tile in Figure 3 can be seen as a geometrical proof of a discrete version of the Pythagorean Theorem. In this discrete version, the right-angled triangles with legs k and $k + 1$ have been replaced by discrete right-angled triangles with both legs of length k, in which the hypotenuse adopts a serrated shape. The area of one of these right-angled triangles coincides with the area of its discrete version.

According to that and to the results on Section 2, we can say that every integer k defines a dense optimal $C_N(1,2k+1)$ graph with diameter k, in which $N = k^2 + (k+1)^2$ nodes. This graph is characterized by four copies of a discrete right-angled triangle with both legs of length k, plus a central unitary square.

Such geometric interpretation, in terms of right-angled triangles, simplifies the study of the distance-related properties of these graphs. For example, to identify the graph's diameter, we fix node 0 as the one located in the unitary central square. As mentioned before, these graphs are vertex-symmetric and thus, the same situation occurs if we select any other node. The remaining nodes are arranged into four adjacent discrete triangles with identical legs of length k (four quadrants with orientations NE, NW, SE and SW), having each one $\frac{k(k+1)}{2}$ nodes. A minimum distance path exists from node 0 to any other node which traverses only one of these four discrete triangles. Within a discrete triangle, the minimum distance between the node located at its right angle and any other node is at most k - 1. Thus, the length of any of these paths is bounded by k, the graph diameter.

It is well known that the diameter of a 2D Torus having N nodes is asymptotically $\sqrt{N}$ (the side of a square of area N). It must be highlighted that, in spite of having the same number of links, the diameter of a $C_N(1,2k+1)$ graph is asymptotically $\frac{\sqrt{N}}{\sqrt{2}}$ (half of the diagonal of the same square of area N). This represents only about 70% of that of a 2D Torus. Conversely, for a given diameter k, a $C_N(1,2k+1)$ graph can arrange k^2 more nodes than its Torus counterpart.



20	2	9	16	23	5	12	19	1	8	15
19	1	8	15	22	4	11	18	0	7	14
18	0	7	14	21	3	10	17	24	6	13
17	24	6	13	20	2	9	16	23	5	12
16	23	5	12	19	1	8	15	22	4	11
15	22	4	11	18	0	7	14	21	3	10
14	21	3	10	17	24	6	13	20	2	9
13	20	2	9	16	23	5	12	19	1	8
12	19	1	8	15	22	4	11	18	0	7
11	18	0	7	14	21	3	10	17	24	6
10	17	24	6	13	20	2	9	16	23	5

Figure 3: Pythagorean Interpretation of a $C_{25}(1,7)$ Graph.

Another important reason for which $C_N(1,2k+1)$ graphs are especially suitable for designing interconnection networks is that they are ideal for implementing diffusive communication models such as packet broadcasting and all-to-all packet interchange. Using again the above geometric characterization we can easily obtain an optimal broadcasting algorithm for dense graphs. We will sketch this algorithm assuming multiport I/O, in which a node can simultaneously communicate with all its neighbours. As in many related algorithms, we can embed a spanning tree into the original graph, whose root is the source node. A spanning tree starting at node 0, will reach its four neighbours in a first step, each one being located at the right-angle of its corresponding discrete triangle. Within a given triangle, packets propagate in two directions, i.e. N and E for the top-right triangle. In order to avoid duplicates, only the nodes located in the first column of a triangle will retransmit packets in both directions, while the other nodes will forward them only within its own row. As this algorithm is optimal, the broadcasting time will depend on *k*. A 2D Torus does not exhibit the regularity of the dense $C_N(1,2k+1)$ graphs in which, from a given node always exist 4*d* different nodes at distance *d*. This combined with the Torus' higher diameter leads to more complex and slower broadcasting algorithms [6].

Related also to another geometrical proof of the Pythagorean theorem, a new layout for $C_N(1,2k+1)$ graphs can be obtained as shown in Figure 4. This representation can be especially suitable for implementing interconnection networks for highly coupled parallel systems because it minimizes the number of wrap-around links. The graph is now seen as two attached squares with sides *k* and *k* + 1. The number of wrap-around links is equal to the semi-perimeter of the resulting polygon, which is 3*k* + 2, reducing by *k* the number of those links needed by the layout depicted in Figure 2.

An important issue when designing an interconnection network is the selection of the most adequate layout of the underlying topology. Minimizing the number of wire's crosses and equalizing the length of all the network links must be goals to be pursued in order to achieve a good network design.

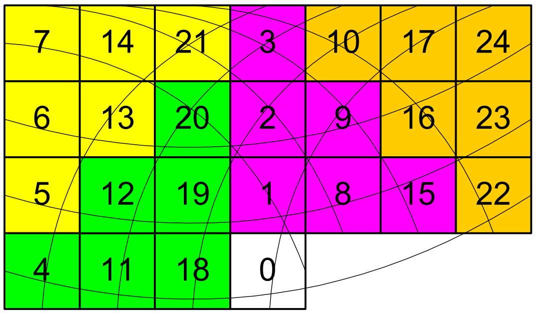


Figure 4: Mesh-connected $C_{25}(1,7)$ Graph with Minimum Number of Wrap-around Links.

The rest of this Section is devoted to introduce a three-dimensional graph transformation that naturally avoids the unbalance between internal mesh-like links and external wrap-around links. When technological advances permit it, our three-dimensional layout should be one of the best choices for implementing an interconnection network based on circulant graphs with both minimal topological and physical distances among nodes.

The nodes of an optimal graph can be placed on the surface of a Torus with radii *R* and *r*, as shown in Figure 5, by using its parametric equations, defined as follows:

$$\begin{cases} x(u,v) = \cos(v)(R - r\cos(u)) \\ y(u,v) = \sin(v)(R - r\cos(u)) \quad u,v \in [0,2\pi] \\ z(u,v) = r\sin(u) \end{cases}$$

If we define, as in [9], $u_j = \frac{2\pi}{N} j$ and $v_j = \frac{2\pi(2k+1)}{N} j$, then every node *j* of the graph, with $j \in \{0,1,\ldots,N-1\}$, corresponds to a point T_j of the Torus surface by using the mapping

$$T_j = \Psi(j) = (x(u_j,v_j), y(u_j,v_j), z(u_j,v_j))$$

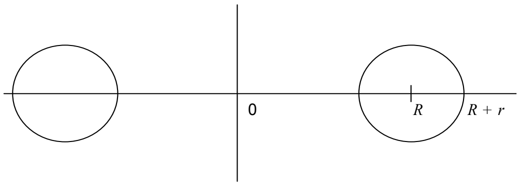


Figure 5: Cut Through Torus Having Radii R and r.

The points $T_j = (t_{x(j)}, t_{y(j)}, t_{z(j)})$, being t_x, t_y, t_z their Cartesian coordinates, are located on the helix

$$T(s;1,2k+1) = (x(u_s,v_s), y(u_s,v_s), z(u_s,v_s)),$$

where $s \in [0, N-1]$. Similarly, the same points $T_j = (t_{x(j)}, t_{y(j)}, t_{z(j)})$, are located on the curve $T(s;\, 2k+1, -1)$, $s \in [0, N-1]$. The intersection points of $T(s;1,2k+1)$ and $T(s;\, 2k+1, -1)$ are the points T_j, in which we locate every node *j* of the graph. Actually, both curves define two disjoint Hamiltonian paths, which are always embedded in this class of graphs. Figure 6 shows a $C_N(1,2k+1)$ graph with diameter $k=3$ and $R=2r$, case in which the length of all the graph edges is the same, being this length the closest value to the Euclidean distance between two neighbour nodes [9].

This embedding of the $C_N(1,2k+1)$ graph on a Torus surface can lead to a 3D network implementation with all the physical links having the same minimum length and never crossing among them. In addition, the underlying graph has the minimal topological distances among all the circulants of degree four. Some designs for futuristic applications as the ones introduced in [10], suggest the use in the few next years of cylindrical structures and free-space optical communications for implementing extreme interconnection networks.

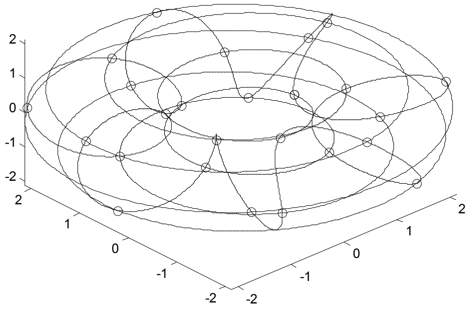


Figure 6: 3D Torus Embedding of the $C_{25}(1,7)$ Graph.

IV. OPTIMAL DENSE CHORDAL RINGS AS A SET OF RINGS

It can be shown that dense optimal $C_N(1,2k+1)$ graphs can be decomposed into a non-disjoint collection of $2N$ rings that preserves the distance-related properties of the original graph. These rings, or circulants of degree two, are also dense and their diameter is the diameter of the circulant in which they are embedded.

In short, the ring decomposition is done as follows (the the proofs are in [13]). In order to obtain this set of rings and avoiding repeated cases in the study, only are considered rings obtained by taking positive steps through jumps $\{1,2k+1\}$ or by taking positive steps through jumps $\{1,-(2k+1)\}$. Therefore, for the rest of the paper, a $(2k+1)$-ring embedded on a $C_N(1,2k+1)$ is a cycle such that, if $\{n_1,n_2,\dots,n_{2k+1}\}$ is its set of nodes, then one of the following assertions must be true:

- $n_{i+1} = n_i + 1 \bmod N$ or $n_{i+1} = n_i + (2k+1) \bmod N$, $\forall i \in \{1,2,\dots,2k+1\}$.
- $n_{i+1} = n_i + 1 \bmod N$ or $n_{i+1} = n_i - (2k+1) \bmod N$, $\forall i \in \{1,2,\dots,2k+1\}$.

For better understanding, we represent a $(2k+1)$-ring as a vector whose coordinates are in the set {0, 1} or in {0,-1}, depending on the type of $(2k+1)$-ring. If we are considering a $(2k+1)$-ring in a $C_N(1,2k+1)$, 0 represents a (positive) step in jump 1, 1 represents a step in $2k+1$ and -1 a step in $-(2k+1)$. For example, the 7-tuple $(1,0,0,1,0,0,1)$ denotes the 7-ring $\{0,7,8,9,16,17,18\}$ in $C_{25}(1,7)$.

Using the above notation, we consider the following sets of $(2k+1)$-rings:

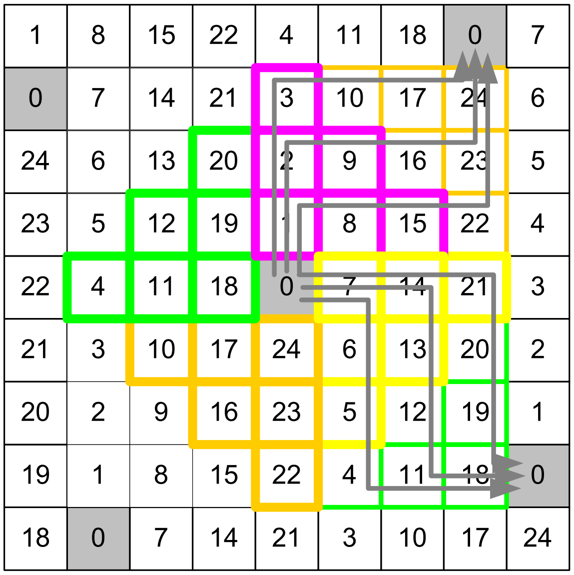
- $A_1^k = \{\overline{\lambda_1}, \overline{\lambda_2}, \dots, \overline{\lambda_k}\} \subset A_1$,where

$$\overline{\lambda_i} = (\overbrace{0,\dots,0}^{i}, \overbrace{1,\dots,1}^{k}, \overbrace{0,\dots,0}^{(k+1)-i}),\quad i = 1, 2, \dots, k.$$

- $A_2^k = \{\overline{\mu_1}, \overline{\mu_2}, \dots, \overline{\mu_k}\} \subset A_2$,where

$$\overline{\mu_i} = (\overbrace{-1,\dots,-1}^{i}, \overbrace{0,\dots,0}^{k}, \overbrace{-1,\dots,-1}^{(k+1)-i}),\quad i = 1, 2, \dots, k.$$

Then, for every node n of $C_N(1,2k+1)$ there exists an element in $A_1 \cup A_2$ containing n. This set of $2k$ rings gives us a complete picture of the graph from node 0. An example of this rings from node 0 in $C_{25}(1,7)$ can be seen in Figure 7.



1	8	15	22	4	11	18	0	7
0	7	14	21	3	10	17	24	6
24	6	13	20	2	9	16	23	5
23	5	12	19	1	8	15	22	4
22	4	11	18	0	7	14	21	3
21	3	10	17	24	6	13	20	2
20	2	9	16	23	5	12	19	1
19	1	8	15	22	4	11	18	0
18	0	7	14	21	3	10	17	24

Figure 7: Characteristic Set of Rings for Node 0 on a $C_{25}(1,7)$ Graph.

Finally, we consider the union of subsets of $(2k+1)$-rings for all the nodes in the graph. It can be shown [13] that this union has $2N$ different elements (each ring from node 0 is repeated exactly k times). Hence, the graph has been decomposed into a set of $2N$ $(2k+1)$-rings, providing a complete picture of the graph.

Then, we can say that, for any integer k, there is a dense $C_{k^2+(k+1)^2}(1,2k+1)$ graph, which can be seen as a collection of $2\left(k^2+(k+1)^2\right)$ dense rings. In this collection, every ring is composed of $2k+1 = k+(k+1)$ nodes. The value of the integer k corresponds to the diameter of both graphs and totally defines both topologies.

In a recent paper of the authors [12], it was also proved that, for any integer k, there is a non-dense $C_{k^2+k^2}(1,2k-1)$ graph, which can be seen as a collection of $2\left(2k^2\right)$ non-dense rings. In this collection, every ring is composed of $2k = k+k$ nodes. The value of the integer k corresponds to the diameter of both graphs and totally defines both topologies.

Finally, we can conclude that some specific members of the family of optimal circulant graphs of degree four are a particular evolution of the traditional ring topology or circulants of degree two. In fact, for $k > 1$, we can see the non-dense optimal Chordal Ring as an evolution from the non-dense ring and the dense optimal Chordal Ring as an evolution from the dense ring. Any of these degree four circulants can be seen as a homogeneous collection of $2N$ rings (or degree two circulants), which conserves the distance properties of the original graph.

V. CONCLUSIONS

In this paper we have presented both an algebraic and a geometrical characterizations of the class of dense optimal circulant graphs of degree four. These graphs are optimal because, for a given diameter, they are the densest ones among all the circulants of degree four. The diameter of a graph in this class is lower, in a factor $\left(1/\sqrt{2} \right)$, than that of its torus counterpart. This can be translated in a performance improvement for interconnection networks based on these optimal circulants.

We have proved that all the dense optimal circulants are isomorphic and we have selected a Chordal Ring as the representative member of this interesting graph class. In addition, we have presented two geometrical characterizations to gain an insight into the topological nature of such Chordal Rings. One of these geometrical views characterizes these graphs by means of four copies of a discrete right-angled triangle with legs *k* and *k*. The other, is a three dimensional characterization which distributes equidistantly the graph nodes over a torus' surface. This provides a methodology to implement three-dimensional networks based on these graphs and using optical free-space communications.

As another by-product of this analysis, we have shown that these dense optimal graphs have a distance-hereditary non-disjoint decomposition into a set of 2*N* dense rings with identical diameter. This property can be successfully exploited to map multiple parallel applications that can be solved in terms of rings. This result can be combined with another recent algebraic characterization of a sister class of these graphs, published in [12]. Both studies allow us to present an infinite number of optimal dense Chordal Rings that are an evolution of the traditional ring topology.

VI. ACKNOWLEDGMENTS

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