# Algebraic Crossover Operators for Permutations

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# Finitely Generated Group

- A group is a set $X$ endowed with an operation $\circ: X \times X \to X$ that:
  - is associative $(x \circ y) \circ z = x \circ (y \circ z) \quad \forall x, y, z \in X$
  - has a neutral element $e \in X$ s.t. $x \circ e = e \circ x = x \quad \forall x \in X$
  - has inverse elements $\exists\, x^{-1}$ s.t. $x \circ x^{-1} = x^{-1} \circ x = e \quad \forall x \in X$
- A group $X$ is finitely generated if there exists a finite generating set $H \subseteq X$ such that every $x \in X$ can be expressed as a finite composition of the generators in $H$, i.e.,

$$x = h_{i_1} \circ h_{i_2} \circ \cdots \circ h_{i_k} \text{ with } h_* \in H$$

- The length of a minimal decomposition of $x$ in terms of H is the weight of $x$, we denote it with $|x|$
- $x \sqsubseteq y$ iff there exists (at least) a minimal decomposition of x that is a prefix of a minimal decomposition of y

# Permutations form a group

- A permutation of $[n] = \{1,2,\dots,n\}$ is a bijective discrete function from $[n]$ to $[n]$, thus it is possible to compose permutations:

$$z = x \circ y \text{ iff } z(i) = x(y(i)) \text{ for } 1 \leq i \leq n$$

- The composition:
  - is associative $(x \circ y) \circ z = x \circ (y \circ z)$
  - has neutral element $e = \langle 1,2,\dots,n \rangle$
  - has inverse elements $x^{-1}(i) = j \text{ iff } x(j) = i$

- The permutations of $[n]$, together with the $\circ$ operation, form a group structure called the symmetric group $\mathcal{S}(n)$

# Permutations form a F.G. group

- Adjacent swap moves as generating set

$$ASW = \{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$$

where, for any $1 \leq i < n$:

$$\sigma_i(j) = \begin{cases} i+1 & \text{if } j = i \\ i & \text{if } j = i+1 \\ j & \text{otherwise} \end{cases}$$

- $(x \circ \sigma_i)$ is the permutation $x$ where the items at positions $i$ and $i+1$ have been swapped

# Cayley graph

- A finitely generated group induces a colored digraph (namely, a Cayley graph) where:
  - $X$ are the vertices
  - there exists an arc $x \rightarrow (x \circ h)$ colored by $h$ for every $x \in X$ and $h \in H \subseteq X$
- Connection between a Cayley graph and a combinatorial search space:
  - $X$ is the set of solutions
  - $H \subseteq X$ represents simple search moves in the space of solutions
- The Cayley graph induces:
  - neighborhood relationships among solutions
  - a distance between solutions (shortest path distance)
  - **a single representation for both solutions and displacements**

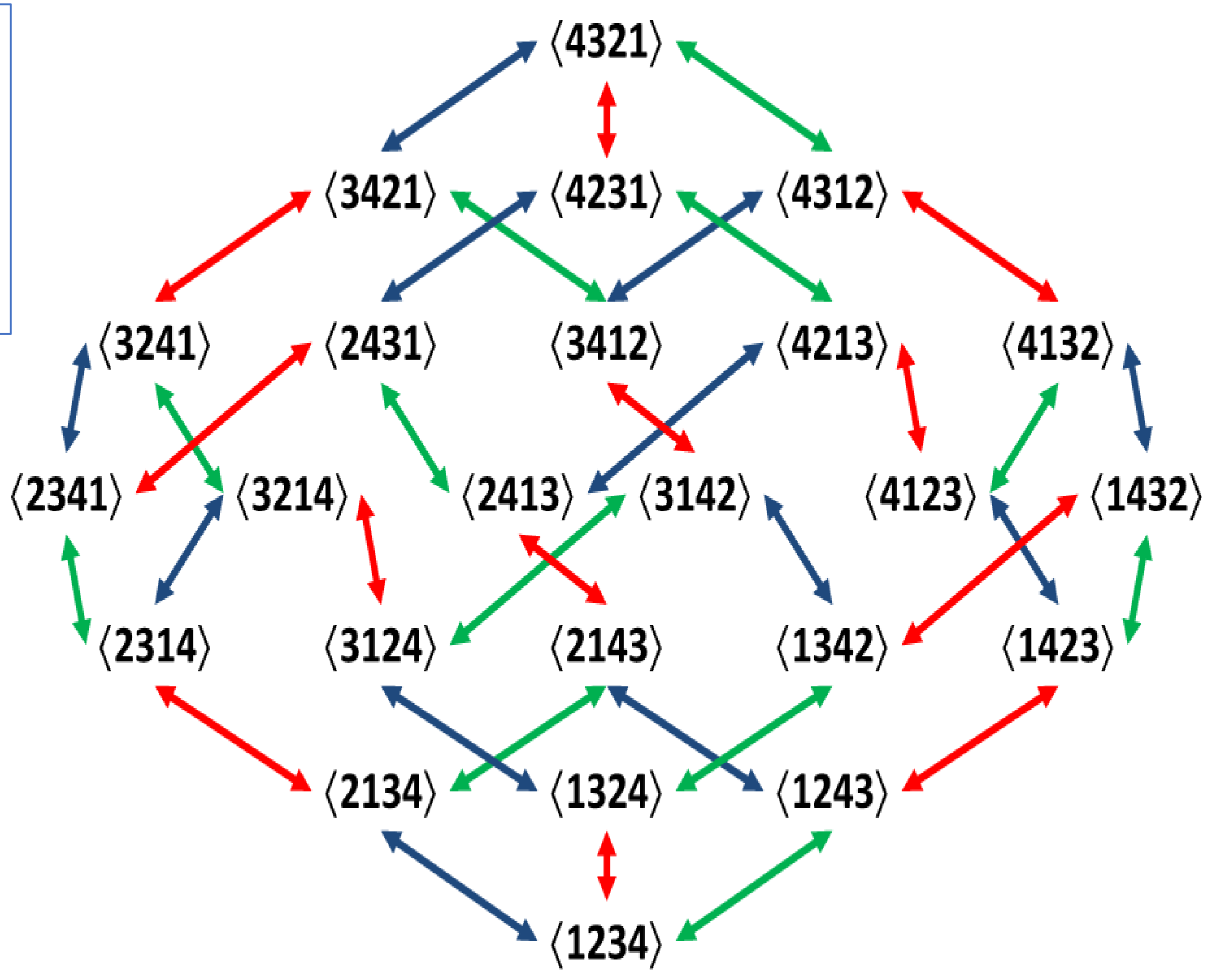
# Cayley graph

**Generators:**

<2134>

<1324>

<1243>



# Vector-like operations

- Paths on the Cayley graph can be encoded by composing the labels on the arcs => paths can be encoded using permutations!
- Permutations encode both «points» and «vectors» in the search space
- Let's define $\oplus$, $\ominus$, $\odot$ in such a way that they work consistently w.r.t. their usual numerical counterparts!

# Sum and Difference of Permutations

- Given two *n*-length permutations *x* and *y*:
  - $x \oplus y := x \circ y$
  - $x \ominus y := y^{-1} \circ x$

- They are geometrically meaningful:
  - $x \oplus y$ is the point reachable starting from point *x* and following the path *y*
  - $x \ominus y$ represents a path connecting the point *y* to the point *x*

- They are algebraically consistent:

$$x = y \oplus (x \ominus y) = y \circ (y^{-1} \circ x) = x$$

# Multiply a permutation by a scalar in [0,1]

- Given a *n*-length permutation *z* and a scalar $a \in [0,1]$
- Let's assume:
  - *z* represents a path
  - $z = \sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_L}$ is a minimal decomposition of $z$ with length $L$
- $a \odot z \coloneqq \sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_k}$ where $k = \lceil a \cdot L \rceil$
- Geometrically, $a \odot z$ is a «truncation» of the path *z*

# How to compute them?

- $\oplus$ and $\ominus$ simply require permutations composition and inversion
- $\odot$ requires an algorithm for computing minimal decomposition(s)
- There can be multiple minimal decompositions
- They can be computed using a «bubble sort»-like algorithm:

  Iteratively choose (and apply) an adjacent swap moving the permutation closer to the identity permutation (the only sorted permutation)

- Two different strategies:
  - <u>RandBS</u>: randomly choose one suitable adjacent swap
  - <u>GreedyBS</u>: choose the best (and suitable) adjacent swap basing on the fitness function

# RandBS and GreedyBS

**function** RANDBS($x \in \mathcal{S}_n$)
  $s \leftarrow \langle\rangle$
  $A \leftarrow \{\sigma_i \in ST : i < i+1 \text{ and } x(i) > x(i+1)\}$
  **while** $A \neq \emptyset$ **do**
    $\sigma \leftarrow$ select a generator from $A$ uniformly at random
    $x \leftarrow x \circ \sigma$
    $s \leftarrow$ Concatenate$(\langle\sigma\rangle, s)$
    $A \leftarrow$ Update$(A, \sigma)$  ▷ $O(1)$ complexity
  **end while**
  **return** $s$
**end function**

**function** GREEDYBS($x \in \mathcal{S}_n, f : \mathcal{S}_n \rightarrow \mathbb{R}$)
  $s \leftarrow \langle\rangle$
  $A \leftarrow \{\sigma_i \in ST : i < i+1 \text{ and } x(i) > x(i+1)\}$
  **while** $A \neq \emptyset$ **do**
    $\sigma \leftarrow$ select $\sigma \in A$ with the best value of $f(x \circ \sigma)$
    $x \leftarrow x \circ \sigma$
    $s \leftarrow$ Concatenate$(\langle\sigma\rangle, s)$
    $A \leftarrow$ Update$(A, \sigma)$  ▷ done in $O(1)$ complexity
  **end while**
  **return** $s$
**end function**

# What can we do with them?

- Algebraic Differential Evolution

- Algebraic Particle Swarm Optimization

- Discretize numerical EAs whose «move rules» are linear combinations of solutions

- … design an algebraic crossover

# Algebraic Crossovers

- We propose 3 classes of algebraic crossovers:
  - Group-based algebraic crossovers
  - Lattice-based algebraic crossovers
  - Hybrid algebraic crossovers

# Group-based Algebraic Crossovers (AXG)

- Reasonably, a crossover between two permutations *x* and *y* should return a permutation *z* that is «in the middle» between *x* and *y*
- The interval [*x*,*y*] can be formally defined in many equivalent ways:

$$[x, y] = \{z \in \mathcal{S}_n \, : \, \exists p \in SP_{x,y} \text{ s.t. } z \text{ appears in } p\},$$

$$[x, y] = \{z \in \mathcal{S}_n \, : \, z \ominus x \sqsubseteq y \ominus x\},$$

$$[x, y] = \{z \in \mathcal{S}_n : d(x, z) + d(z, y) = d(x, y)\}.$$

$$[x, y] = \{z \in \mathcal{S}_n \, : \, D(x, z) \subseteq D(z, y)\}.$$

- A group-based algebraic crossover operator AXG can be abstractly defined as any operator which, given two permutations *x* and *y*, returns a permutation *z* = AXG(*x*,*y*) such that $z \in [x, y]$

# A property on pairwise precedences of items

- *z* = AXG(*x*,*y*)

- AXG is precedence-respectful:

  *z* contains all the common precedences between *x* and *y*

- AXG transmits precedences:

  all the common precedences of *z* come from *x* or *y*

  (no new precedence is generated)

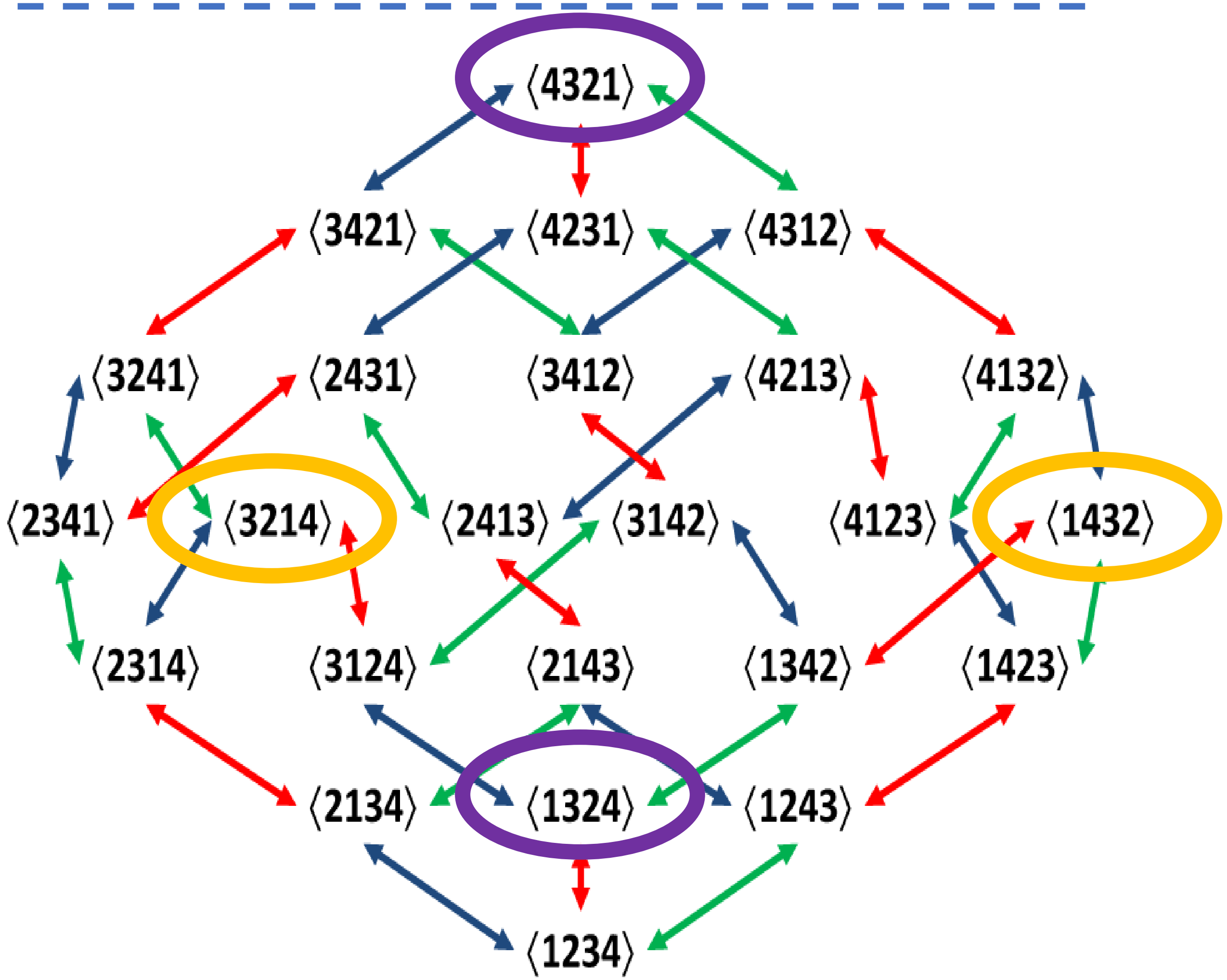
# How to compute AXG?

- Ideally:
  - $z = x \oplus a \odot (y \ominus x)$, with $a \in [0,1]$
  - Different strategies basing on the value of *a* and the $\odot$ computation strategy
- Practically:
  - We enumerate all the permutations in a shortest path from *x* to *y* by running RandBS (R) or GreedyBS (G) on $y \ominus x$
  - We select one permutation in the path: a random one (R), the middle one (T), the best one (B)
- 6 different implementations:
  - AXG-RR, AXG-RT, AXG-RB
  - AXG-GR, AXG-GT, AXG-GB

# The partial order $\sqsubseteq$ is a Lattice

- Meet: $z = x \wedge y$ iff
  1. $z \sqsubseteq x$ and $z \sqsubseteq y$
  2. $z$ is the «longest» permutation with prop.1

- Join: $z = x \vee y$ iff
  1. $x \sqsubseteq z$ and $y \sqsubseteq z$
  2. $z$ is the «shortest» permutation with prop.1



# Meet and Join as Crossover Operators

**function** AXL-MEET($x \in \mathcal{S}_n, y \in \mathcal{S}_n$)
    $x' \leftarrow x^{-1}$
    $y' \leftarrow y^{-1}$
    $z \leftarrow e$
    $A \leftarrow \{\sigma_i \in ST : x'(i) > x'(i+1) \text{ and } y'(i) > y'(i+1)\}$
    **while** $A \neq \emptyset$ **do**
        $\sigma \leftarrow$ select a generator from$A$
        $x' \leftarrow x' \circ \sigma$
        $y' \leftarrow y' \circ \sigma$
        $z \leftarrow z \circ \sigma$
        $A \leftarrow \text{Update2}(A, \sigma)$ ▷ done in $O(1)$ complexity
    **end while**
    **return** $z$
**end function**

AXL-Join exploits the «De Morgan»-like property:

$$x \vee y = \left(x^R \wedge y^R\right)^R$$

# Hybrid Algebraic Crossovers

- Let AXG be any group-based algebraic crossover
- Let $m$ = AXL-Meet($x$,$y$)
- Let $j$ = AXL-Join($x$,$y$)

- An hybrid algebraic crossover AXH is defined as

$$\text{AXH}(x,y) := \text{AXG}(m,j)$$

- 6 hybrid alg. crossovers: one for each AXG crossover

# AXH-* are more explorative than AXG-*

- AXH crossovers produce an offspring $z \in [x \vee y, x \wedge y]$
- $[x, y] \subseteq [x \vee y, x \wedge y]$
- It is possible to introduce precedences not appearing in the parents

# Experimental Setting

- Benchmark instances from LOP, PFSP, QAP, TSP
- Comparison against 7 popular permutation crossover from literature: PMX, OX1, OX2, CX, AP, POS, ER

- Three scenarios:
  - Randomly generated permutations
  - Local optima permutations
  - Crossovers embedded in standard steady-state GA

    (population size = 50, random selection, crossover prob = 1, mutation prob = 0.05, $\mu$+1 replacement)

# Experimental Results

AVERAGE RANKS IN THE RANDOM EXPERIMENT

| Crossover | LOP | PFSP | QAP | TSP | Overall |
|---|---|---|---|---|---|
| *AXH-GB* | **1.12** | **1.55** | **1.33** | **1.12** | **1.28** |
| *AXG-GB* | 3.04 | 2.74 | 1.70 | 5.40 | 3.22 |
| *AXH-RB* | 7.39 | 3.92 | 5.83 | 3.25 | 5.10 |
| *AXG-RB* | 8.10 | 5.10 | 5.76 | 7.78 | 6.69 |
| *AXH-GT* | 3.90 | 11.91 | 5.79 | 5.50 | 6.78 |
| *AXH-GR* | 5.52 | 10.14 | 6.62 | 5.39 | 6.92 |
| *AXG-GT* | 5.94 | 9.15 | 5.97 | 10.75 | 7.95 |
| *AXG-GR* | 7.75 | 8.77 | 6.10 | 10.55 | 8.29 |
| *AXL-Meet* | 12.54 | 11.21 | 16.53 | 8.50 | 12.20 |
| *AXH-RR* | 14.46 | 14.14 | 15.38 | 10.96 | 13.74 |
| *AXH-RT* | 14.86 | 13.70 | 14.51 | 11.88 | 13.74 |
| *AXL-Join* | 16.53 | 15.79 | 16.29 | 8.44 | 14.26 |
| *ER* | 14.09 | 13.42 | 14.35 | 15.58 | 14.36 |
| *OX1* | 14.35 | 13.54 | 14.31 | 15.74 | 14.49 |
| *CX* | 14.41 | 13.69 | 14.33 | 15.75 | 14.55 |
| *OX2* | 14.45 | 13.60 | 14.44 | 15.75 | 14.56 |
| *AP* | 14.52 | 13.70 | 14.34 | 15.72 | 14.57 |
| *PMX* | 14.42 | 13.74 | 14.36 | 15.74 | 14.57 |
| *POS* | 14.46 | 13.74 | 14.38 | 15.70 | 14.57 |
| *AXG-RR* | 14.53 | 13.74 | 14.35 | 15.73 | 14.59 |
| *AXG-RT* | 14.61 | 13.70 | 14.33 | 15.75 | 14.60 |

AVERAGE RANKS IN THE LOCAL OPTIMA EXPERIMENT

| Crossover | LOP | PFSP | QAP | TSP | Overall |
|---|---|---|---|---|---|
| *AXG-GB* | **1.28** | **1.31** | **1.22** | **1.29** | **1.28** |
| *AXG-RB* | 1.78 | 1.82 | 1.91 | 1.78 | 1.82 |
| *AXG-GR* | 3.93 | 5.47 | 8.44 | 7.20 | 6.26 |
| *CX* | 8.31 | 6.12 | 4.43 | 10.52 | 7.35 |
| *PMX* | 9.38 | 6.84 | 6.44 | 6.80 | 7.37 |
| *AXG-GT* | 5.08 | 7.05 | 9.36 | 8.41 | 7.48 |
| *OX2* | 8.80 | 6.93 | 9.12 | 10.82 | 8.92 |
| *POS* | 8.80 | 7.00 | 9.04 | 10.87 | 8.93 |
| *OX1* | 10.22 | 10.84 | 11.97 | 4.08 | 9.28 |
| *AXH-GB* | 12.84 | 12.64 | 4.40 | 9.50 | 9.85 |
| *AP* | 6.03 | 7.11 | 16.33 | 16.58 | 11.51 |
| *AXG-RR* | 6.45 | 9.51 | 15.54 | 14.60 | 11.53 |
| *AXH-RB* | 15.49 | 12.48 | 9.93 | 11.65 | 12.39 |
| *AXG-RT* | 8.31 | 11.34 | 16.36 | 16.32 | 13.08 |
| *AXH-GT* | 14.19 | 16.23 | 9.70 | 14.90 | 13.76 |
| *AXH-GR* | 16.36 | 16.48 | 10.25 | 14.37 | 14.37 |
| *ER* | 20.60 | 19.67 | 16.27 | 3.99 | 15.13 |
| *AXH-RT* | 16.34 | 15.73 | 16.83 | 20.08 | 17.25 |
| *AXL-Meet* | 19.22 | 19.11 | 18.10 | 13.95 | 17.60 |
| *AXH-RR* | 17.61 | 17.36 | 17.37 | 19.38 | 17.93 |
| *AXL-Join* | 19.97 | 19.94 | 18.01 | 13.87 | 17.95 |

AVERAGE RANKS IN THE GA EXPERIMENT

| Crossover | LOP | PFSP | QAP | TSP | Overall |
|---|---|---|---|---|---|
| *PMX* | 9.33 | 2.33 | **1.33** | 6.67 | **4.92** |
| *POS* | 6.67 | **2.00** | 2.33 | 9.33 | 5.08 |
| *AX-Comb* | 3.33 | 7.67 | 8.00 | **2.33** | 5.33 |
| *OX2* | 5.67 | 6.00 | 4.00 | 8.00 | 5.92 |
| *CX* | **2.33** | 3.00 | 7.67 | 11.67 | 6.17 |
| *AXG-GB* | 4.00 | 5.67 | 7.33 | 8.67 | 6.42 |
| *OX1* | 8.67 | 5.00 | 8.00 | 4.00 | 6.42 |
| *AXG-RB* | 5.33 | 7.33 | 11.33 | 10.00 | 8.50 |
| *AXG-GR* | 3.33 | 9.00 | 10.00 | 14.00 | 9.08 |
| *AXG-RR* | 8.33 | 8.33 | 14.00 | 13.33 | 11.00 |
| *AXH-RB* | 14.33 | 13.33 | 7.33 | 10.00 | 11.25 |
| *AXG-GT* | 9.00 | 12.67 | 7.67 | 18.00 | 11.84 |
| *AP* | 13.67 | 13.33 | 16.00 | 5.00 | 12.00 |
| *AXH-GB* | 15.33 | 16.67 | 11.00 | 11.00 | 13.50 |
| *AXG-RT* | 14.67 | 13.33 | 12.33 | 18.00 | 14.58 |
| *AXH-RR* | 16.67 | 15.00 | 15.00 | 11.67 | 14.59 |
| *AXH-GR* | 14.33 | 18.33 | 14.67 | 12.33 | 14.92 |
| *AXH-GT* | 16.33 | 15.67 | 16.00 | 16.00 | 16.00 |
| *AXH-RT* | 18.67 | 15.33 | 16.00 | 15.67 | 16.42 |
| *ER* | 20.00 | 20.00 | 21.33 | 11.67 | 18.25 |
| *AXL-Join* | 21.33 | 21.00 | 21.00 | 17.67 | 20.25 |
| *AXL-Meet* | 21.67 | 22.00 | 20.67 | 18.00 | 20.59 |

AXG crossovers loss diversity very quickly when inside a GA
AXH crossovers slow down the diversity loss but only slightly

# Conclusion and Future Work

- What's new:

  Group and lattice structures of the search space exploited to build crossovers and derive some properties of them

- Issues to address:

  Need to better balance intensification and diversification

- Other future work:
  - AXG operators using different generating sets (EXC and INS)
  - AXL operators using semi-lattice
  - Build a clever GA using AX operators
  - Precedence properties may help to design a good recombination for LOP

Thanks!!!