Search Moves in the Local Optima Networks of Permutation Spaces

The QAP case

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Motivations and Goal

- One of the achieved objectives of Fitness Landscape Analysis (FLA) is: "estimate how many search moves need to be performed in order to escape an attraction basin" …

- … but FLA does not identify which moves have to be performed!
- Our goal: identify the "escaping moves"

Outline

- Quadratic Assignment Problem
- Local Optima Network
- Algebraic framework for Evolutionary Computation
- Qualitative analysis of the “escaping moves”
- Future lines of research

Quadratic Assignment Problem (QAP)

- There are n factories and n cities.
- A distance a_{ij} is specified for each pair of cities.
- A flow b_{ij} is specified for each pair of factories.
- The problem is to assign all factories to different cities with the goal of minimizing the sum of the distances multiplied by the corresponding flows.

$$C(\pi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{\pi_i \pi_j}$$

Fitness Landscape (FL)

A FL is a triplet (X,N,f) where:

- X is the set of solutions
  (all the permutations of n items in QAP)
- N is a neighborhood structure among the solutions
  (exchange neighborhood in QAP)
- f is a fitness function
  (the QAP objective function)

Local Optima Network (LON)

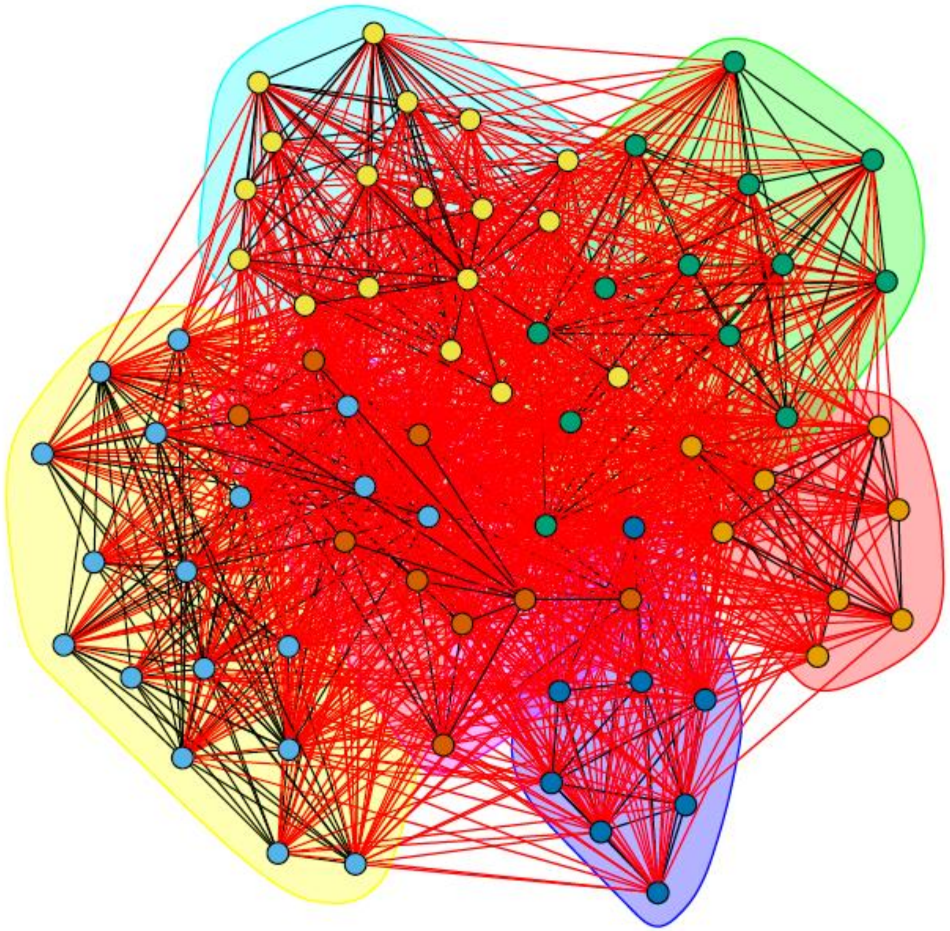
A LON is a graph extracted from a given FL by using a (best-improvement) hill-climber *hc* where:

- the nodes are the local optima of the given FL
- there is an (escape) edge e_{ij} between LO_i and LO_j if a solution x exists such that
  $$dist(x, LO_i) \leq D \text{ and } hc(x) = LO_j$$
- the edge e_{ij} has weight $w_{ij} = v_{ij} / \Sigma_j \, v_{ij}$ where
  $$v_{ij} = \#\{\, x \in X \mid dist(x, LO_i) \leq D \text{ and } hc(x) = LO_j \,\}$$

COMMUNITIES OF OPTIMA IN LONS

- LONs are complex networks which can be studied with methods of network science

- LONs can have a clustered structure, thus the local optima can be divided in communities



ALGEBRAIC STRUCTURE OF THE PERMUTATION SPACE

- A permutation of $[n] = \{1,2,\dots,n\}$ is a bijective discrete function from $[n]$ to $[n]$, thus it is possible to invert and compose permutations:

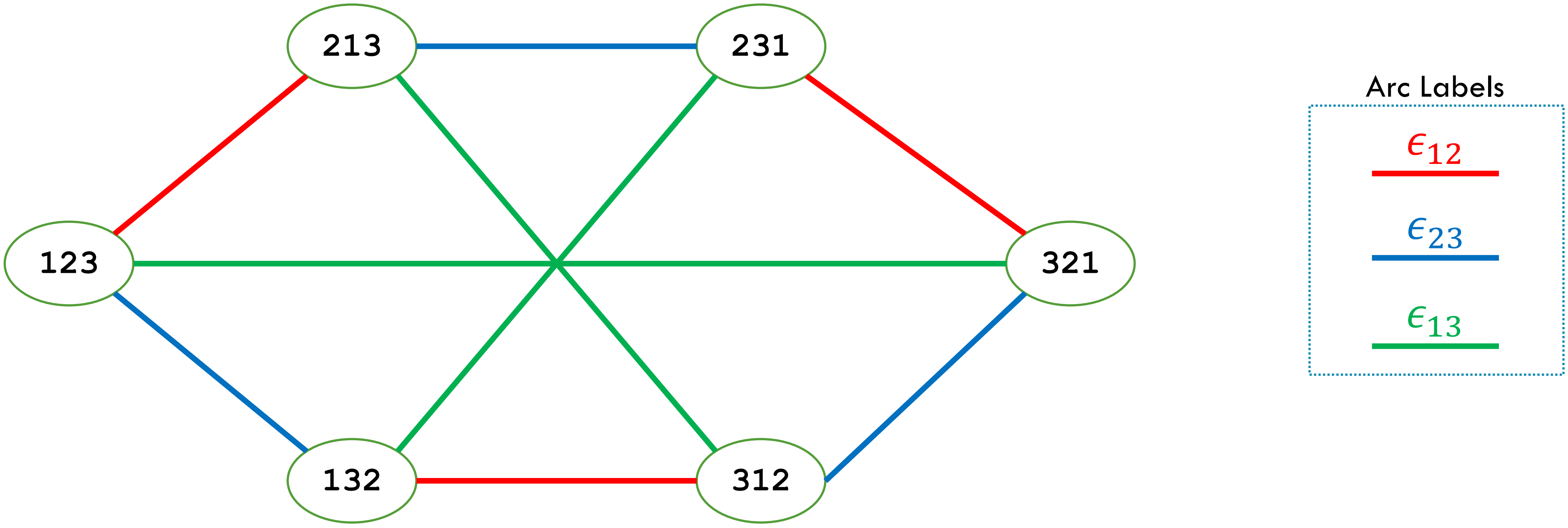
$$\sigma = \pi \circ \rho \text{ iff } \sigma(i) = \pi(\rho(i)) \text{ for } 1 \leq i \leq n$$

- Permutations of $[n]$ form the symmetric group $\mathcal{S}(n)$
- $\mathcal{S}(n)$ is finitely generated by the exchange permutations ϵ_{ij} s.t.

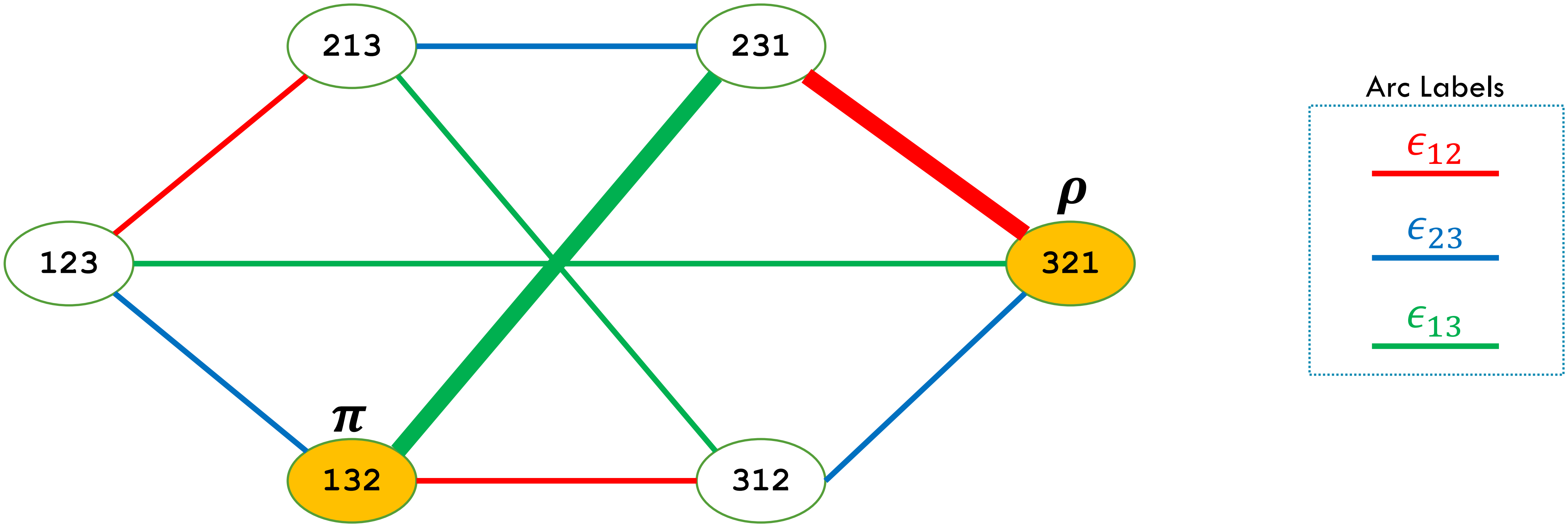
$$\epsilon_{ij}(k) = \begin{cases} k & \text{if } k \neq i \text{ and } k \neq j \\ j & \text{if } k = i \\ i & \text{if } k = j \end{cases}$$

- Given any $\pi \in \mathcal{S}(n)$, $\pi \circ \epsilon_{ij}$ corresponds to exchanging the items at positions *i* and *j* in the permutation π

The Cayley Graph

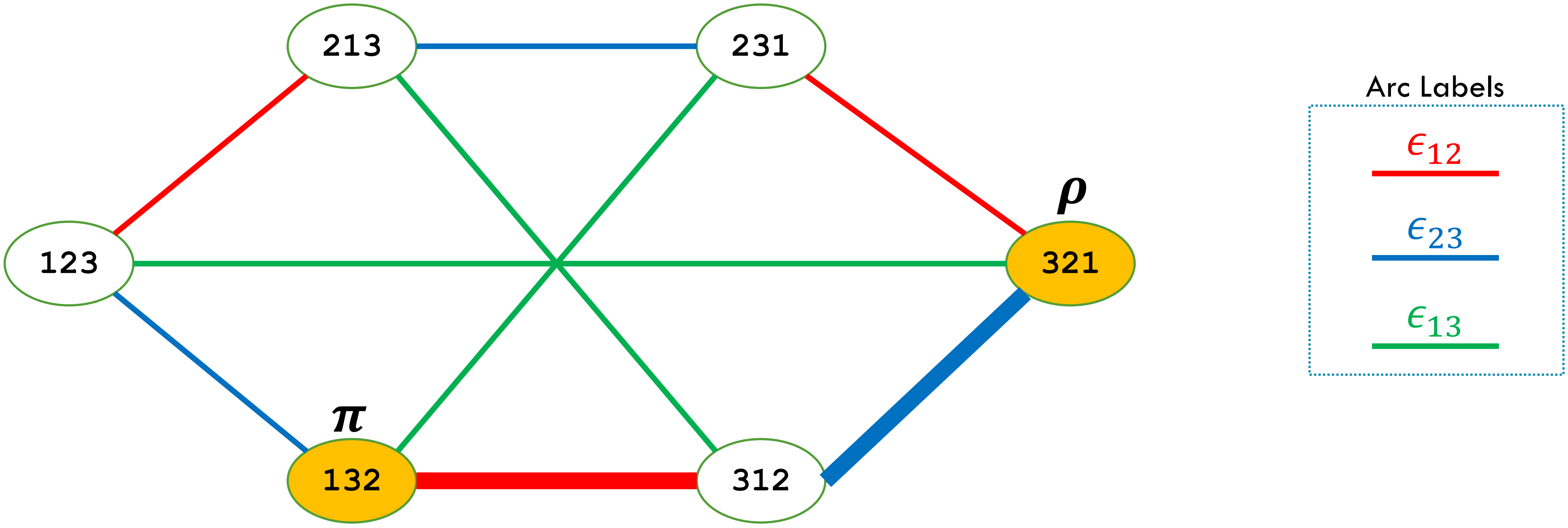


Differences between Permutations



$$\rho \ominus \pi = \epsilon_{13} \circ \epsilon_{12}$$

DIFFERENCES BETWEEN PERMUTATIONS



$$\rho \ominus \pi = \epsilon_{13} \circ \epsilon_{12} = \epsilon_{12} \circ \epsilon_{23}$$

Differences between Permutations

- All the paths connecting π to ρ in the Cayley graph are all the possible factorizations of $\rho \ominus \pi$
- Given any pair of permutations π, ρ their difference is

$$\rho \ominus \pi = \pi^{-1} \circ \rho$$

- … but the factorizations of $\rho \ominus \pi$ indicate the sequences of pairs of positions to exchange …
- … while we want the sequences of pairs of items to exchange!!!

ITEMS TO EXCHANGE (AND NOT POSITIONS!!!)

- A permutation is a bijection from positions to items

- We can exchange two generic **items** i and j from π as follows

$$\left(\pi^{-1} \circ \epsilon_{ij}\right)^{-1} = \epsilon_{ij} \circ \pi$$

- The sequences of pairs of **items** to be exchanged for moving from π to ρ correspond to all the possible factorizations of

$$\pi^{-1} \ominus \rho^{-1} = \rho \circ \pi^{-1}$$

Compact Representation of the Exchanges

- Given two permutations, the number of alternative (shortest) paths connecting them is exponential in their distance
- We want to identify the pairs of items to exchange independently of where they appear in the factorizations
- We use the cycle decomposition of a permutation

$$\begin{matrix} 12345678 \\ \langle 26745831 \rangle \end{matrix} \Rightarrow (1268)(37)(4)(5)$$

$$(1,2),(1,6),(1,8),(2,6),(2,8),(6,8),(3,7)$$

Weigh Exchanges by Importance (1/2)

- Given two permutations, the exchanges appearing in multiple (shortest) paths between them are more important
- If the two permutations are local optima, an exchange appearing in a large number of (shortest) paths connecting them is more useful for escaping a basin of attraction
- Let consider that a factorization in terms of exchanges can be obtained by iteratively exchanging two items belonging to the same cycle:
  - The cycle breaks into two (smaller) cycles
  - The identity permutation is the only one with *n* cycles (of length 1)

Weigh Exchanges by Importance (2/2)

- Pair of items in shorter cycles (w.r.t. all the other cycles) appear in a large number of factorizations
- Closer are two items in a cycle, more are the factorizations where they appear

- Some approximated formulae and tabulations in the paper (obtained by considering a recursive variant of our factorization algorithm)

The Experimental Analysis

- LONs of few QAP real-like instances (thanks to Sebastien Verel)
- Clustered by means of two community finding algorithms (R package *igraph*)

- Intra-Community Analysis: are there more relevant exchanges for moving between local optima of the same community?
- Inter-Community Anlysis: are there more relevant exchanges for moving between local optima of different communities?

The AnalyzeExchanges Algorithm

Algorithm 2 Computation of the importance of the exchanges in order to move among a set of permutations

```
function ANALYZEEXCHANGES(S ⊂ 𝒮_n × 𝒮_n)
    Z_ij ← 0 for all 1 < i < j ≤ n
    for all pairs of permutations π, ρ ∈ S do
        Compute the cycles decomposition of π^{-1} ⊖ ρ^{-1}
        for all cycles c of π^{-1} ⊖ ρ^{-1} do
            for all pairs of items i, j ∈ c do
                Z_ij ← Z_ij + Q_k(ϵ_ij)
            end for
        end for
    end for
    return Z
end function
```

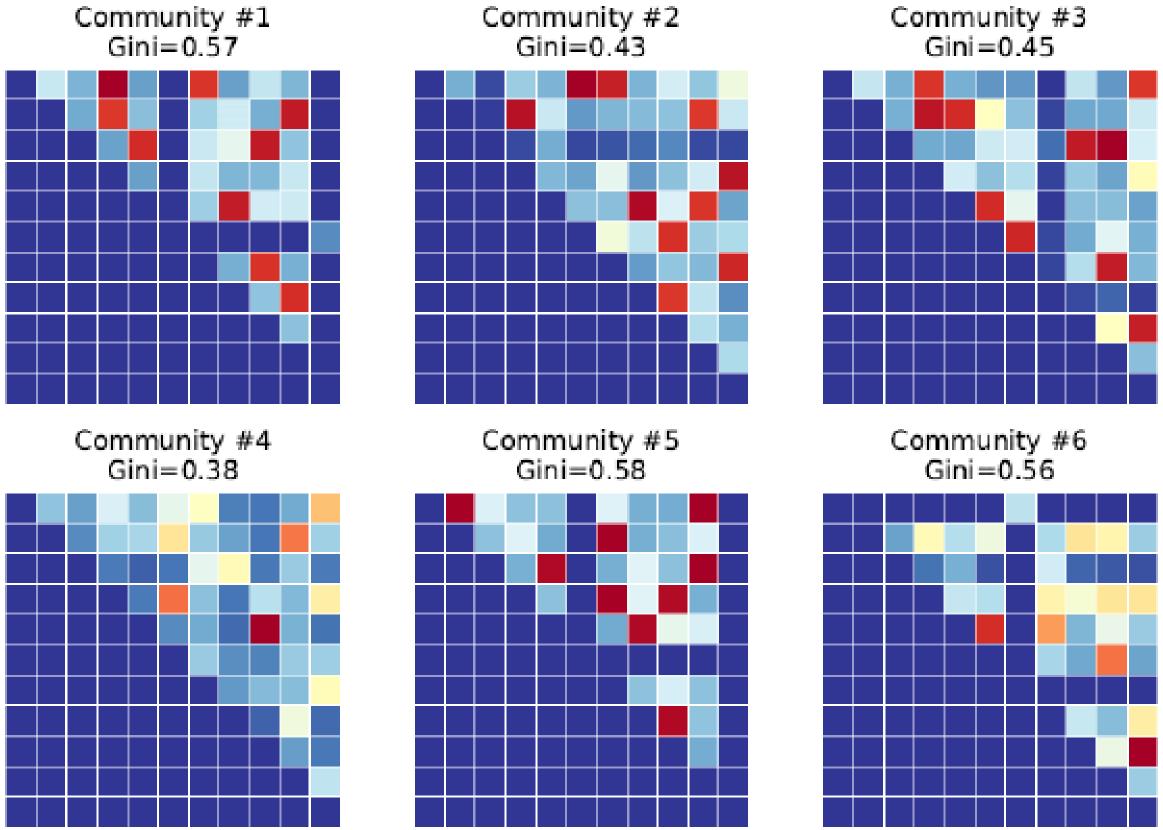
Exchange Importance (pointing to $Q_k(\epsilon_{ij})$)

- **INPUT:**
  - (Intra-Comm. An.) set of local optima in a same community
  - (Inter-Comm. An.) set of local optima in different communities

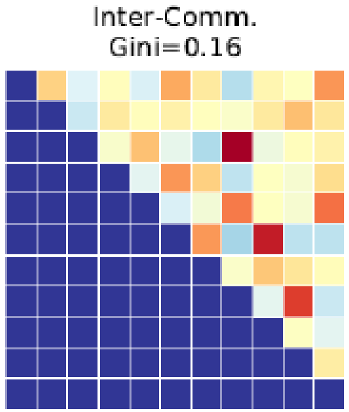
- **OUTPUT:** a (triangular) matrix such that Z_{ij} measures the relevance of ϵ_{ij} as escaping move

Heatmaps for KCso11rl-1 (walktrap)

Intra-Community Analysis



Inter-Community Analysis



GINI INDEXES ON THE Z-VALUES

- The Gini Index is a measure of statistical dispersion (popular in economy)
  - 0 on uniform distributions, 1 on degenerate distributions
  - in our scenario, 1 is impossible (due to the constraints among permutation items)
  - 0.5 has been empirically observed to produce a «concentrated» distribution

Table 2: Gini Indexes for every clustered LON

Instance	Community Finding Algorithm	#Communities	Intra-Community Gini Index	Inter-Community Gini Index
KCso11rl-1	FastGreedy	6	0.56 ± 0.14	0.16
KCso11rl-1	WalkTrap	6	0.50 ± 0.09	0.16
KCso11rl-2	FastGreedy	6	0.46 ± 0.11	0.12
KCso11rl-2	WalkTrap	2	0.26 ± 0.15	0.14
KCso11rl-3	FastGreedy	8	0.44 ± 0.10	0.18
KCso11rl-3	WalkTrap	6	0.41 ± 0.13	0.17
KCso11rl-4	FastGreedy	5	0.57 ± 0.08	0.16
KCso11rl-5	WalkTrap	5	0.58 ± 0.08	0.16

Conclusions

- Real-like QAP instances look to have «preferred» search moves that allow to move across basins of attraction belonging to the same community
- The same does not look to be true for basins of attraction belonging to different communities

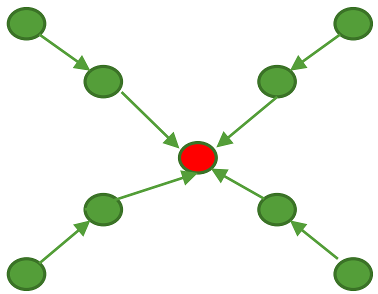
- This analysis shows that the Algebraic Framework for EC can be useful for fitness landscape analyses

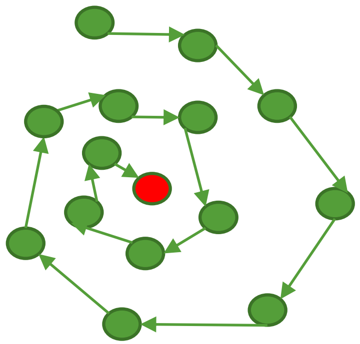
Future Works

- ❑Experiment with larger QAP instances and sampled LONs
- ❑Consider other permutation problems

- ❑Other applications of the Algebraic Framework to FLA:

  «Vortexity» index to discern the following type of basin of attractions





THANKS FOR YOUR ATTENTION!