Proceedings of the
**International Conference on Complex Systems**
Nashua, NH, 21–26 Sept. 1997.

Y. Bar-Yam (ed.), New England Complex Systems Institute (1997) .
Also on-line in the *InterJournal*.

# Generic mechanisms for hierarchies

Didier Sornette*
LPMC, CNRS and Université de Nice-Sophia Antipolis, 06108 Nice Cedex 2, France
and IGPP and ESS, Box 951567, Los Angeles, CA 90095-1567, USA

Common to essentially all complex systems is the existence of a hierarchical organization, with transfers across the different levels from the smallest to the largest ones and vice-versa. Here, we propose that hierarchies are generically formed dynamically due to a spontaneous breakdown of continuous scale invariance into discrete scale invariance. We present a short introduction to the concept of discrete scale invariance and how it leads to complex critical exponents (or dimensions), i.e. to the log-periodic corrections to scaling. Proposed illustrations are diffusion-limited-aggregation clusters, rupture in heterogeneous systems, earthquakes, animals (a generalization of percolation) among several other systems. We refer to several papers and a recent review for further information, especially on the use for predictions.

## 1 Introduction

Notwithstanding the extraordinary large variety of disciplines addressed in this conference, at least one simple concept was recurrent in most if not all presentations, namely the existence of a hierarchical structure. Indeed, in most complex systems, many length scales are present but they do not play an equivalent role. Rather, there are discrete levels in a global hierarchy, each level being described by a set of laws and tools. Examples are found in the weather (dust devils, tornadoes, cyclones, large scale motions), in the tectonic crust (joints, small faults, main faults, plate boundaries), in proteins (primary, secondary, tertiary, quaternary structures), in physiology (molecules, cells, tissues, systems), in the brain (neurons, functional regions, lobes, hemispheres), in economy (traders, companies, countries, currency blocks), in society (individual, family, local social group, town, region, country), etc.

However, the question of the origin of this hierarchy is poorly understood probably due to the fact that many detailled mechanisms come into play. Here, we would like to propose a robust "meta-mechanism" according to which hierarchies can naturally emerge from the *partial* breakdown of the most natural symmetry in systems with many length scales, namely the symmetry of continuous scale invariance.

The partial breakdown of continuous scale invariance corresponds to *discrete scale invariance*. We have urged [1] that generic field theories are non-unitary and, as a consequence, have operators with complex dimensions. Complex dimensions or exponents is synonymous to log-periodic behavior, and therefore to a discrete scale invariance.

After their initial suggestion as formal solutions of renormalization group equations in the seventies, complex exponents have been studied in the eighties in relation to various problems of physics embedded in hierarchical systems. Only recently has it been realized that discrete scale invariance and its associated complex exponents may appear "spontaneously" in euclidean systems, i.e. without the need for a pre-existing hierarchy. We briefly list the known mechanisms for the spontaneous generation of discrete scale invariance and situations where complex exponents have been found. This is done in order to provide a basis for a better fundamental understanding of discrete scale invariance. The main motivation to study discrete scale invariance and its signatures is that it provides new insights in the underlying mechanisms of scale invariance. It may also be very interesting for prediction purposes.

We refer to [2] for a recent review.

## 2 What is 'Discrete Scale Invariance'

Let us first recall what is the concept of (continuous) scale invariance: in a nutshell, it means reproducing itself on different time or space scales. More precisely, an observable $\mathcal{O}$ which depends on a "control" parameter $x$ is scale invariant under the arbitrary change $x \to \lambda x$ [1], if there is a number $\mu(\lambda)$ such that

$$\mathcal{O}(x) = \mu \mathcal{O}(\lambda x) \quad . \tag{1}$$

Eq.(1) defines a homogeneous function and is encountered in the theory of critical phenomena, in turbulence, etc. Its solution is simply a power law

$$\mathcal{O}(x) = C x^{\alpha} \ , \tag{2}$$

with

$$\alpha = -\frac{\log \mu}{\log \lambda} \ , \tag{3}$$

which can be verified directly by insertion. Power laws are the hallmark of scale invariance as the ratio $\frac{\mathcal{O}(\lambda x)}{\mathcal{O}(x)} = \lambda^{\alpha}$ does not depend on $x$, i.e. the relative value of the observable at two different scales only depend on the *ratio* of

*Supported in part by NSF EAR96-15357 and NSF EAR97-06488

[1]Here, we implicitely assume that a change of scale leads to a change of control parameter as in the renormalization group formalism. More directly, $x$ can itself be a scale.

the two scales [2]. This is the fundamental property that associates power laws to scale invariance, self-similarity, and criticality (Criticality refers to the state of a system which has scale invariant properties. The critical state is usually reached by tuning a control parameter as in liquid-gas and paramagnetic-ferromagnetic phase transitions. Many driven extended out-of-equilibrium systems seem also to exhibit a kind of dynamical criticality, that has been coined "self-organized criticality".

Discrete scale invariance (DSI) is a weaker kind of scale invariance according to which the system or the observable obeys scale invariance as defined above only for specific choices of $\lambda$ (and therefore $\mu$), which form in general an infinite but countable set of values $\lambda_1, \lambda_2, ...$ that can be written as $\lambda_n = \lambda^n$. $\lambda$ is the fundamental scaling ratio. This property can be qualitatively seen to encode a *lacunarity* of the fractal structure.

Note that, since $x \to \lambda x$ and $\mathcal{O}(x) \to \mu\mathcal{O}(\lambda x)$ is equivalent to $y = \log x \to y + \log\lambda$ and $\log\mathcal{O}(y) \to \log\mathcal{O}(y + \log\lambda) + \log\mu$, a scale transformation is simply a translation of $\log x$ leading to a translation of $\mathcal{O}$. Continuous scale invariance is thus the same as continuous translational invariance expressed on the logarithms of the variables. DSI is then seen as the restriction of the continuous translational invariance to a *discrete* translational invariance: $\log\mathcal{O}$ is simply translated when translating $y$ by a multiple of a fundamental "unit" size $\log\lambda$. Going from continuous scale invariance to DSI can thus be compared with (in logarithmic scales) going from the fluid state to the solid state in condensed matter physics! In other words, the symmetry group is no more the full set of translations but only those which are multiple of a fundamental discrete generator.

# 3 Properties

## 3.1 Signature of DSI

The hallmark of scale invariance is the existence of power laws. The signature of DSI is the presence of power laws with *complex* exponents $\alpha$ which manifests itself in data by log-periodic corrections to scaling.

## 3.2 Importance and usefulness of DSI

Log-periodic structures in data indicate that the system and/or the underlying physical mechanisms have characteristic length scales. This is extremely interesting as this provides important constraints on the underlying physics. Indeed, simple power law behaviors are found everywhere, as seen from the explosion of the concepts of fractals, criticality and self-organized-criticality. For instance, the power law distribution of earthquake energies which is known as the Gutenberg-Richter law can be obtained by many different mechanisms and a variety of models and is thus extremely limited in constraining the underlying physics. Its usefulness as a modelling constraint is even doubtful, in contradiction to the common belief held by physicists on the importance of this power law. In contrast, the presence of log-periodic features would teach us that important physical structures, that would be hidden in the fully scale invariant description, existed.

## 3.3 Prediction

It is important to stress the practical consequence of log-periodic structures. For prediction purposes, it is much more constrained and thus reliable to fit a part of an oscillating data than a simple power law which can be quite degenerate especially in the presence of noise. This remark has been used and is vigorously investigated in several applied domains, such as earthquakes, rupture prediction and financial crashes.

# 4 Mechanisms leading to DSI and Examples

We list below the examples discussed in the litterature and refer to [2] for further informations.

- Built-in geometrical hierarchy;
- Programming and number theory;
- Diffusion in anisotropic quenched random lattices;
- Cascade of ultra-violet instabilities: growth processes and rupture;
- Cascades of sub-harmonic bifurcations in the transition to chaos;
- Animals;
- Quenched disordered systems;
- The bronchial tree;
- Turbulence;
- Titius-Bode law;
- Gravitational collapse and black hole formation;
- Rate of escape from stable attractors;
- Interface crack tip stress singularity;
- Eigenfunctions of the Laplace transform.

The references below refer to the work carried out in our team. Ref.[2] provides an extensive list of references covering most of the litterature on this subject.

[2] This is only true for a function of a single parameter. Homogeneous functions of several variables take a more complex form than (1).

# References


[1] Saleur H. and D. Sornette, Complex exponents and log-periodic corrections in frustrated systems, *J.Phys.I France* **6** (1996), 327–355.

[2] Sornette, D., Discrete scale invariance and complex dimensions, *Physics Reports*, in press (1997) (http://xxx.lanl.gov/abs/cond-mat/9707012).

[3] Sornette, D. and C.G. Sammis, Complex critical exponents from renormalization group theory of earthquakes : Implications for earthquake predictions, *J.Phys.I France* **5** (1995), 607–619 .

[4] Sammis, C.G., D. Sornette and H. Saleur, Complexity and earthquake forecasting, Reduction & Predictability of Natural Disasters, eds. J. Rundle, Turcotte D.L. and W. Klein, Santa Fe Institute Studies in the Sciences of Complexity, vol. XXV (Addison-Wesley, 1995), 143–156.

[5] Anifrani, J.-C., C. Le Floc'h, D. Sornette and B. Souillard, Universal Log-periodic correction to renormalization group scaling for rupture stress prediction from acoustic emissions, *J.Phys.I France* **5** (1995), 631–638.

[6] Sornette D., A. Johansen, A. Arnéodo, J.-F. Muzy and H. Saleur, Complex fractal dimensions describe the internal hierarchical structure of DLA, *Phys. Rev. Lett.* **76** (1996), 251–254.

[7] Sornette D., A. Johansen, and J.-P. Bouchaud, Stock market crashes, Precursors and Replicas, *J.Phys.I France* **6** (1996), 167-175.

[8] Saleur H., C.G. Sammis and D. Sornette, Renormalization group theory of earthquakes, *Nonlinear Processes in Geophysics* **3**, (1996), 102–109.

[9] Ouillon G., D.Sornette, A. Genter and C. Castaing, The imaginary part of rock jointing, *J. Phys. I France* **6** (1996), 1127–1139.

[10] A. Johansen, D.Sornette, H. Wakita, U. Tsunogai, W.I. Newman and H. Saleur, Discrete scaling in earthquake precursory phenomena : evidence in the Kobe earthquake, Japan, *J.Phys.I France* **6** (1996) 1391–1402.

[11] Saleur H., C.G. Sammis and D. Sornette, Discrete scale invariance, complex fractal dimensions and log-periodic corrections in earthquakes, *Journal of Geophysical Research-Solid Earth* **101** (1996), 17661–17677.

[12] Huang, Y., G. Ouillon, H. Saleur and D. Sornette, Spontaneous generation of discrete scale invariance in growth models, *Phys. Rev. E* **55** (1997) 6433–6447.

[13] Huang, Y., H. Saleur, C.G. Sammis and D. Sornette, Precursors, aftershocks, criticality and self-organized criticality, Europhys. Lett. in press (http://xxx.lanl.gov/abs/cond-mat/9612065)

[14] Jögi, P., D. Sornette and M. Blank, Fine structure and complex exponents in power law distributions from random maps, *Phys. Rev. E*, in press (1997) (http://xxx.lanl.gov/abs/cond-mat/9708220).