ANALISIS MATEMATICO

1.- Resolver la siguiente ecuación diferencial:

$$y' + 4y = y^2 (\operatorname{sen}x + \cos x)$$

2.- Buscar un factor integrante de la forma: $\mu = \mu\left(x^2 + y^2\right)$ para la siguiente ecuación diferencial:

$$\left(x^3 + xy^2 - y\right)dx + \left(y^3 + x^2 y + x\right)dy = 0$$

3.- Resolver el siguiente sistema de ecuaciones diferenciales:

$$\begin{cases} \dfrac{dx}{dt} = -2x - 3y \\ \dfrac{dy}{dt} = -3x - 2y + 2e^{2t} \end{cases}$$

4.- Resolver la siguiente integral definida:

$$\int_0^{\infty} e^{-2t} \frac{\operatorname{sen}^2 t}{t} dt$$

5.- Calcular:

$$L^{-1}\left[\frac{s+3}{\left(s^2 + 2s + 2\right)^2}\right]$$

6.- Dada la función:

$$f(x) = \begin{cases} 2x & 0 \le x \le \dfrac{1}{2} \\ 2(1-x) & \dfrac{1}{2} < x \le 1 \end{cases}$$

a) Obtener el desarrollo en serie de Fourier en términos de coseno.
b) Desarrollar en términos de seno.

Solución

1º.- Es una ecuación de Bernoulli.

Dividiendo entre y^2 tenemos:

$$\frac{y'}{y^2}+\frac{4}{y}=\text{senx}+\cos x,\text{haciendo}\frac{1}{y}=z\Rightarrow \text{haciendo}\frac{1}{y}=z\Rightarrow z'=-\frac{y'}{y^2}$$

$-z'+4z=\text{senx}+\cos x\Rightarrow$ ecuación lineal.

Resuelta de dos formas:

a) $-r+4=0\Rightarrow r=4\Rightarrow z_H=A\,e^{4x}$, admitirá una particular:

$$z_p=B\cos x+C\,\text{senx};z'_p=-B\,\text{senx}+C\cos x$$

$$-z'_p+4z_p=B\,\text{senx}-C\cos x+4B\cos x+4C\,\text{senx}=\text{senx}+\cos x\Rightarrow$$

$$\Rightarrow\begin{cases}B+4C=1\\-C+4B=1\end{cases}\Rightarrow B=\frac{5}{17},C=\frac{3}{17}.$$

$$z=A\,e^{4x}+\frac{5}{17}\cos x+\frac{3}{17}\text{senx};z=\frac{1}{y}\Rightarrow y=\frac{17}{5\cos x+3\text{senx}+Ke^{4x}}$$

b) Haciendo z=u.v tenemos:

$$-(u'v+uv')+4uv=\text{senx}+\cos x.$$

$$u(-v'+4v)-u'v=\text{senx}+\cos x;\ \text{haciendo}\ (-v'+4v)=0\Rightarrow\frac{dv}{v}=4\,dx\Rightarrow$$

$$\ln v=4x\Rightarrow v=e^{4x}.$$

$$-u'e^{4x}=\text{senx}+\cos x\Rightarrow u=-\int e^{-4x}(\text{senx}+\cos x)dx,\text{ciclíca}$$

$$\int e^{-4x}(\text{senx}+\cos x)dx=-\frac{1}{4}e^{-4x}(\text{senx}+\cos x)+\frac{1}{4}\int e^{-4x}(\cos x-\text{senx})dx=$$

$$-\frac{1}{4}e^{-4x}(\text{senx}+\cos x)+\frac{1}{4}\left[-\frac{1}{4}e^{-4x}(\cos x-\text{senx})+\frac{1}{4}\int e^{-4x}(-\text{senx}-\cos x)dx\right]\Rightarrow$$

$$\left(1+\frac{1}{16}\right)\int e^{-4x}(\text{senx}+\cos x)dx=e^{-4x}\left\{\left(-\frac{1}{4}+\frac{1}{16}\right)\text{senx}+\left(-\frac{1}{4}-\frac{1}{16}\right)\cos x\right\}\Rightarrow$$

$$\int e^{-4x}(\text{senx}+\cos x)dx=\frac{16}{17}e^{-4x}\left(\frac{-3}{16}\text{senx}+\frac{-5}{16}\cos x\right)=-\frac{1}{17}e^{-4x}(3\text{senx}+5\cos x)$$

$$u=\frac{1}{17}e^{-4x}(3\text{senx}+5\cos x)+C$$

$$\frac{1}{y}=z=uv=\frac{1}{17}\left(3\text{senx}+5\cos x+17Ce^{4x}\right)\Rightarrow y=\frac{17}{3\text{senx}+5\cos x+Ke^{4x}}.$$

2º.- $P=x^3+xy^2-y\Rightarrow P_y=2xy-1;Q=y^3+x^2y+x\Rightarrow Q_x=2xy+1.$

$$(\mu P)_y = (\mu Q)_x \Rightarrow \mu_y P + \mu P_y = \mu_x Q + \mu Q_x$$

$$\text{llamando}: x^2 + y^2 = t \begin{cases} \mu_x = \dfrac{\partial \mu}{\partial t}.\dfrac{\partial t}{\partial x} = \mu'.2x \\ \mu_y = \dfrac{\partial \mu}{\partial t}.\dfrac{\partial t}{\partial y} = \mu'.2y \end{cases}$$

$$\mu'.2y.P + \mu P_y = \mu'.2x.Q + \mu Q_x \Rightarrow \mu'(2y.P - 2x.Q) = \mu(Q_x - P_y)$$

$$\frac{\mu'}{\mu} = \frac{(Q_x - P_y)}{(2y.P - 2x.Q)} = \frac{2}{2(y.P - x.Q)} = \frac{1}{(x^3y + xy^3 - y^2) - (xy^3 - x^3y + x^2)} =$$

$$\frac{1}{-(x^2 + y^2)} = -\frac{1}{t} \Rightarrow \frac{d\mu}{\mu} = -\frac{1}{t}dt \Rightarrow \ln\mu = -\ln t \Rightarrow \mu = \frac{1}{t} = \frac{1}{x^2 + y^2}.$$

3º.- Derivando la primera ecuación:

$$x'' = -2x' - 3y' = -2x' - 3(-3x - 2y + 2e^{2t}) = -2x' + 9x + 6y - 6e^{2t}$$

$$y = \frac{1}{3}(-2x - x') \text{ sustituyendo } x'' = -2x' + 9x - 4x - 2x' - 6e^{2t} \Rightarrow$$

$$x'' + 4x' - 5x =' -6e^{2t};\ r^2 + 4r - 5 = 0 \Rightarrow r = -2 \pm \sqrt{4+5} = \begin{cases} 1 \\ -5 \end{cases}$$

$$x_H = Ae^{-5t} + Be^{t}.$$

$$x_p = Ce^{2t} \Rightarrow x' = 2Ce^{2t} \Rightarrow x'' = 4Ce^{2t}.$$

$$x''_p + 4x'_p - 5x_p = (4C + 8C - 5C)e^{2t} = -6e^{2t} \Rightarrow C = -\frac{6}{7}.$$

$$x = Ae^{-5t} + Be^{t} - \frac{6}{7}e^{2t}.$$

$$y = \frac{1}{3}(-2x - x') = \frac{1}{3}\left(-2Ae^{-5t} - 2Be^{t} + \frac{12}{7}e^{2t} + 5Ae^{-5t} - Be^{t} + \frac{12}{7}e^{2t}\right) =$$

$$= Ae^{-5t} - Be^{t} + \frac{8}{7}e^{2t}.$$

4º.- $\displaystyle\int_0^\infty e^{-2t}\frac{\operatorname{sen}^2 t}{t}dt = \lim_{s\to 2} L\left[\frac{\operatorname{sen}^2 t}{t}\right].$

Como el $\displaystyle\lim_{t\to 0}\frac{\operatorname{sen}^2 t}{t} = L\acute{}H\hat{o}p = \lim_{t\to 0}\frac{2\operatorname{sen}t\cos t}{1} = 0.$

$$L\left[\frac{sen^2t}{t}\right]=\int_s^\infty f(u)du;\ \text{siendo } f(u)=L\left[sen^2t\right].$$

$$sen^2t=\frac{1}{2}(1-\cos 2t)\Rightarrow L\left[sen^2t\right]=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right).$$

$$L\left[\frac{sen^2t}{t}\right]=\int_s^\infty \frac{1}{2}\left(\frac{1}{u}-\frac{u}{u^2+4}\right)du=\frac{1}{2}\left(\ln u-\frac{1}{2}\ln(u^2+4)\right)\Bigg|_s^\infty=$$

$$\frac{1}{4}\ln\left(\frac{u^2}{u^2+4}\right)\Bigg|_s^\infty=\frac{1}{4}\left\{\lim_{u\to\infty}\ln\left(\frac{u^2}{u^2+4}\right)-\ln\left(\frac{s^2}{s^2+4}\right)\right\}=$$

$$=\frac{1}{4}\left[\ln 1+\ln\left(\frac{s^2+4}{s^2}\right)\right]=\frac{1}{4}\left[\ln\left(\frac{s^2+4}{s^2}\right)\right].$$

$$\text{luego } \int_0^\infty e^{-2t}\frac{sen^2t}{t}dt=\lim_{s\to 2}L\left[\frac{sen^2t}{t}\right]=\frac{1}{4}\left[\ln\left(\frac{4+4}{4}\right)\right]=\frac{1}{4}\ln 2.$$

5º.- $L^{-1}\left[\frac{s+3}{(s^2+2s+2)^2}\right]=L^{-1}\left[\frac{(s+1)+2}{\left((s+1)^2+1\right)^2}\right]=1^a\,\text{trasl}=e^{-t}L^{-1}\left[\frac{s+2}{\left(s^2+1\right)^2}\right].$

De dos formas:

a) $f(s)=-\frac{1}{2}\left(\frac{1}{s^2+1}\right)\Rightarrow f'(s)=\frac{s}{(s^2+1)^2}$. Aplicando la propiedad:

$$L^{-1}[f'(s)]=-tL^{-1}[f(s)]=-t\left(-\frac{1}{2}sent\right)=\frac{1}{2}t\,sent.$$

$$L^{-1}\left[\frac{1}{(s^2+1)^2}\right]=\text{dividir por "s"}=L^{-1}\left[\frac{\frac{s}{(s^2+1)^2}}{s}\right]=\int_0^t F(z)dz.$$

Siendo F(z)= $L^{-1}\left[\frac{s}{(s^2+1)^2}\right]=\frac{1}{2}z\,sen\,z.$

$$\int_0^t \frac{1}{2}z\,senz\,dz=\frac{1}{2}\left[-z\cos z+\int \cos z\,dz\right]_0^t=\frac{1}{2}(-z\cos z+senz)\Big|_0^t=\frac{1}{2}(-t\cos t+sent).$$

$$e^{-t}L^{-1}\left[\frac{s+2}{\left(s^2+1\right)^2}\right]=e^{-t}\left[\frac{1}{2}t\,sent+2\left(\frac{1}{2}(-t\cos t+sent)\right)\right]=e^{-t}\left(\frac{1}{2}t\,sent-t\cos t+sent\right)$$

b) $L^{-1}\left[\frac{s+2}{\left(s^2+1\right)^2}\right]=L^{-1}\left[\left(\frac{s}{\left(s^2+1\right)^2}\right)+\left(\frac{2}{\left(s^2+1\right)^2}\right)\right]=L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right]+2L^{-1}\left[\frac{s}{\left(s^2+1\right)^2}\right].$

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right]=L^{-1}\left[\frac{s}{(s^2+1)}\right]*L^{-1}\left[\frac{1}{(s^2+1)}\right]=\cos t * \operatorname{sen} t=\int_0^t \cos u \operatorname{sen}(t-u)\,du=$$

$$\int_0^t \cos u(\operatorname{sen} t\cos u-\operatorname{sen} u\cos t)du=\operatorname{sen} t\int_0^t \cos^2 u\,du-\cos t\int_0^t \cos u\operatorname{sen} u\,du=$$

$$\operatorname{sen} t\int_0^t \frac{1}{2}(1+\cos 2u)du-\cos t\int_0^t \frac{1}{2}\operatorname{sen} 2u\,du=\frac{1}{2}\left\{\operatorname{sen} t\left[u+\frac{\operatorname{sen} 2u}{2}\right]_0^t-\cos t\left[\frac{-\cos 2u}{2}\right]_0^t\right\}=$$

$$\frac{1}{2}\left\{\operatorname{sen} t\left[t+\frac{\operatorname{sen} 2t}{2}\right]+\cos t\left[\frac{\cos 2t}{2}-\frac{1}{2}\right]\right\}=\frac{1}{2}\left[t\operatorname{sen} t-\frac{1}{2}\cos t+\frac{1}{2}(\operatorname{sen} 2t\operatorname{sen} t+\cos 2t\cos t)\right]=$$

$$\frac{1}{2}\left[t\operatorname{sen} t-\frac{1}{2}\cos t+\frac{1}{2}\cos(2t-t)\right]=\frac{1}{2}t\operatorname{sen} t.$$

$$L^{-1}\left[\frac{1}{(s^2+1)^2}\right]=L^{-1}\left[\frac{1}{(s^2+1)}\right]*L^{-1}\left[\frac{1}{(s^2+1)}\right]=\operatorname{sen} t * \operatorname{sen} t=\int_0^t \operatorname{sen} u \operatorname{sen}(t-u)\,du=$$

$$\int_0^t \operatorname{sen} u(\operatorname{sen} t\cos u-\operatorname{sen} u\cos t)\,du=\operatorname{sen} t\int_0^t \operatorname{sen} u\cos u\,du-\cos t\int_0^t \operatorname{sen}^2 u\,du=$$

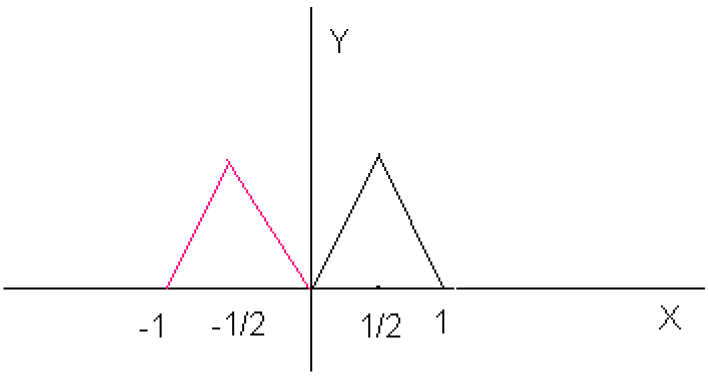
$$\operatorname{sen} t\int_0^t \frac{\operatorname{sen} 2u}{2}du-\cos t\int_0^t \frac{1-\cos 2u}{2}du=\frac{1}{2}\left\{\operatorname{sen} t\left[\frac{-\cos 2u}{2}\right]_0^t-\cos t\left[u-\frac{\operatorname{sen} 2u}{2}\right]_0^t\right\}=$$

$$\frac{1}{2}\left\{-\frac{1}{2}\operatorname{sen} t\cos 2t+\frac{1}{2}\operatorname{sen} t-t\cos t+\frac{1}{2}\cos t\operatorname{sen} 2t\right\}=\frac{1}{2}\left[\frac{1}{2}\operatorname{sen} t-t\cos t+\frac{1}{2}(\operatorname{sen} 2t\cos t-\operatorname{sen} t\cos 2t)\right]=$$

$$\frac{1}{2}\left(\frac{1}{2}\operatorname{sen} t-t\cos t+\frac{1}{2}\operatorname{sen}(2t-t)\right)=\frac{1}{2}(\operatorname{sen} t-t\cos t).$$

$$e^{-t}L^{-1}\left[\frac{s+2}{(s^2+1)^2}\right]=e^{-t}\left[\frac{1}{2}t\operatorname{sen} t+2\left(\frac{1}{2}(\operatorname{sen} t-t\cos t)\right)\right]=e^{-t}\left[\frac{1}{2}t\operatorname{sen} t+\operatorname{sen} t-t\cos t\right].$$

6º.- a) Ya es par, no se necesita hacer la extensión (doblar el periodo).



$$2l = 1 \Rightarrow l = \frac{1}{2}.$$

$$a_0 = \frac{2}{\frac{1}{2}} \int_0^{\frac{1}{2}} 2x\,dx = 4x^2\Big|_0^{\frac{1}{2}} = 4.\frac{1}{4} = 1.$$

$$a_k = \frac{2}{\frac{1}{2}} \int_0^{\frac{1}{2}} 2x\cos\frac{k\pi x}{\frac{1}{2}}\,dx = 4\int_0^{\frac{1}{2}} 2x\cos 2k\pi x\,dx = 8\left(\frac{x\,\mathrm{sen}\,2k\pi x}{2k\pi} - \frac{1}{2k\pi}\int \mathrm{sen}\,2k\pi x\,dx\right)_0^{\frac{1}{2}} =$$

$$= \frac{8}{2k\pi}\left(x\,\mathrm{sen}\,2k\pi x + \frac{1}{2k\pi}\cos 2k\pi x\right)_0^{\frac{1}{2}} = \frac{8}{2k\pi}\left(\frac{1}{2}\mathrm{sen}\,k\pi + \frac{1}{2k\pi}(\cos k\pi - 1)\right) =$$

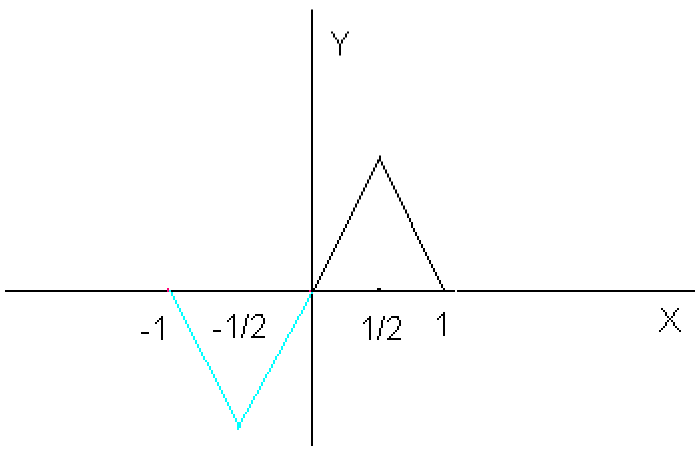
$$\frac{8}{2k\pi}\left(\frac{1}{2k\pi}(\cos k\pi - 1)\right) = \frac{8}{(2k\pi)^2}(\cos k\pi - 1).$$

$$(\cos k\pi - 1) = \begin{cases} 0 & k \text{ par} \\ -2 & k \text{ impar} \end{cases}$$

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(\frac{(2n-1)\pi x}{\frac{1}{2}}\right) = f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\left(2(2n-1)\pi x\right)$$

siendo x punto de continuidad.

b) Hay que hacer la extensión a impar $\Rightarrow$ doblar el periodo y calcular los b_k.



$2l = 2 \Rightarrow l = 1.$

$$b_k = \frac{2}{1}\int_0^1 f(x)\operatorname{sen}\frac{k\pi x}{1}dx = 2\left\{\int_0^{\frac{1}{2}} 2x\operatorname{sen}k\pi x\,dx + \int_{\frac{1}{2}}^{1} 2(1-x)\operatorname{sen}k\pi x\,dx\right\} =$$

$$4\left\{\left[-\frac{x\cos k\pi x}{k\pi} + \frac{1}{k\pi}\int \cos k\pi x\,dx\right]_0^{\frac{1}{2}} + \left[-\frac{(1-x)\cos k\pi x}{k\pi} + \frac{1}{k\pi}\int(-1)\cos k\pi x\,dx\right]_{\frac{1}{2}}^{1}\right\} =$$

$$\frac{4}{k\pi}\left\{\left[-x\cos k\pi x + \frac{\operatorname{sen}k\pi x}{k\pi}\right]_0^{\frac{1}{2}} + \left[(x-1)\cos k\pi x - \frac{1}{k\pi}\operatorname{sen}k\pi x\right]_{\frac{1}{2}}^{1}\right\} =$$

$$\frac{4}{k\pi}\left[-\frac{1}{2}\cos\frac{k\pi}{2} + \frac{1}{k\pi}\operatorname{sen}\frac{k\pi}{2} - \frac{\operatorname{sen}k\pi}{k\pi} - (-\frac{1}{2})\cos\frac{k\pi}{2} + \frac{1}{k\pi}\operatorname{sen}\frac{k\pi}{2}\right] = \frac{4}{(k\pi)^2}2\operatorname{sen}\frac{k\pi}{2} =$$

$$\frac{8}{(k\pi)^2}\operatorname{sen}\frac{k\pi}{2}.$$

$$\operatorname{sen}\frac{k\pi}{2} \Rightarrow \begin{cases} 0 & k \text{ par} \\ \pm 1 & k \text{ impar} \begin{cases} k = 1 \Rightarrow 1 \\ k = 3 \Rightarrow -1 \\ k = 5 \Rightarrow 1 \\ k = 7 \Rightarrow -1 \\ . \\ . \end{cases} \end{cases}$$

$$f(x) = \frac{8}{(\pi)^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\operatorname{sen}(2n-1)\pi x \qquad \text{(x punto de continuidad).}$$