Análisis Matemático

Resolver las siguientes ecuaciones diferenciales:

1º $(2x - y)dx + (x + 5y - 11)dy = 0.$

2º $y(2xy + 1)dx + x(1 + 2xy - x^3y^3)dy = 0.$

Buscando un factor integrante $\mu = \mu(x.y)$.

3º $x^2y'' - xy' - 3y = 5x^3 \ln x$.

4º Hallar: $\int_0^\infty e^{-t} \frac{\text{sen}t \, \text{sen}4t}{t} dt$

5º Hallar: $L^{-1}\left[\frac{e^{-s}}{(s^2 + 2s + 5)^2}\right]$

6º Sea: $f(x) = \begin{cases} x^2 & 0 < x < 1 \\ 1 & 1 \leq x < 2 \end{cases}$

a) Obtener el desarrollo en serie de Fourier.
b) Desarrollar en serie de senos.

1º Resolviendo el sistema $\begin{cases} 2x - y = 0 \\ x + 5y - 11 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}$ haciendo el cambio:

$x=X+1, y=Y+2 \Rightarrow y' = Y'$ nos queda la ecuación homogénea:

$Y' = \frac{Y - 2X}{X + 5Y}$, con el cambio $Y=u.X$ tenemos:

$$u'X + u = \frac{u - 2}{1 + 5u} \Rightarrow u'X = \frac{-5u^2 - 2}{1 + 5u} \Rightarrow \frac{1 + 5u}{5u^2 + 2} du = -\frac{dX}{X}$$

$$\int \frac{1}{2\left(\frac{5}{2}u^2 + 1\right)} du + \frac{1}{2}\ln(5u^2 + 2) = -\ln X + \ln C$$

$$\frac{\sqrt{2}}{2\sqrt{5}} \operatorname{arctg}\left(\frac{\sqrt{5}}{\sqrt{2}}u\right) + \frac{1}{2}\ln(5u^2 + 2) = -\ln X + \ln C$$

$$\frac{\sqrt{2}}{\sqrt{5}} \operatorname{arctg}\left(\frac{\sqrt{5}}{\sqrt{2}}u\right) + \ln(5u^2 + 2) = -2\ln X + 2\ln C$$

$$\left(\frac{(5u^2 + 2)X^2}{K}\right) = e^{-\frac{\sqrt{2}}{\sqrt{5}}\operatorname{arctg}\left(\frac{\sqrt{5}}{\sqrt{2}}u\right)}$$

Deshaciendo los cambios:

$$5(y - 2)^2 + 2(x - 1)^2 = k\, e^{-\frac{\sqrt{2}}{\sqrt{5}}\operatorname{arctg}\left(\frac{\sqrt{5}(y-2)}{\sqrt{2}(x-1)}\right)}.$$

2º $P = 2xy^2 + y \Rightarrow P_y = 4xy + 1 \qquad Q = x + 2x^2y - x^4y^3 \Rightarrow Q_x = 1 + 4xy - 4x^3y^3$.

$$\mu_y P + \mu P_y = \mu_x Q + \mu Q_x \Rightarrow x\mu' P + \mu P_y = y\mu' Q + \mu Q_x \Rightarrow (xP - yQ)\mu' = \mu(Q_x - P_y)$$

$$\frac{\mu'}{\mu} = \frac{Q_x - P_y}{xp - yQ} = \frac{-4x^3y^3}{x^4y^4} = \frac{-4}{xy} \Rightarrow \ln\mu = -4\ln(xy) \Rightarrow \mu = \frac{1}{x^4y^4}.$$

La ecuación:

$$\frac{2xy^2 + y}{x^4y^4}dx + \frac{x + 2xy^2 - x^4y^3}{x^4y^4}dy = 0$$ ya es exacta.

$$\left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx+\left(\frac{1}{x^3y^4}+\frac{2}{x^2y^3}-\frac{1}{y}\right)dy=0$$

$$F(x,y)=-\frac{1}{x^2y^2}-\frac{1}{3x^3y^3}+\alpha(y)$$

$$\frac{\partial F}{\partial y}=\frac{2}{x^2y^3}+\frac{1}{x^3y^4}+\alpha'(y)=\left(\frac{1}{x^3y^4}+\frac{2}{x^2y^3}-\frac{1}{y}\right)\Rightarrow\alpha'(y)=-\frac{1}{y}$$

$$\alpha(y)=-\ln y$$

$$-\frac{1}{x^2y^2}-\frac{1}{3x^3y^3}-\ln y=Cte\Rightarrow\frac{1}{x^2y^2}+\frac{1}{3x^3y^3}+\ln y=\ln k\Rightarrow$$

$$\frac{k}{y}=e^{\frac{1}{x^2y^2}+\frac{1}{3x^3y^3}}=e^{\frac{3xy+1}{3x^3y^3}}\Rightarrow k=ye^{\frac{3xy+1}{3x^3y^3}}$$

3º Con el cambio:

$$x=e^t\Rightarrow\begin{cases}y'_x=y'_t.e^{-t}\\ y''_x=(y''_t-y'_t).e^{-2t}\end{cases}$$

Nos queda la ecuación:

$$y''_t-2y'_t-3y=5te^{3t};r^2-2r-3=0\Rightarrow r=-1,3$$

$$y_H=Ae^{-t}+Be^{3t};$$

$$y_p=t(C+Dt)e^{3t}$$

$$y'_p=\left(3Dt^2+(3C+2D)t+C\right)e^{3t}$$

$$y''_p=\left(9Dt^2+(9C+12D)t+6C+2D\right)e^{3t}$$

Obligando a cumplir la ecuación nos da C=-5/16, D=5/8.

$$y=Ae^{-t}+Be^{3t}+\frac{5t}{16}\left(-1+2t\right)e^{3t}=\frac{A}{x}+Bx^3+\frac{5\ln x}{16}(2\ln x-1)x^3$$

4º

$$\int_0^\infty e^{-t}\frac{\operatorname{sen}t\operatorname{sen}4t}{t}dt=\lim_{s\to1}L\left[\frac{\operatorname{sen}t\operatorname{sen}4t}{t}\right];$$

$$\lim_{t\to0}\left[\frac{\operatorname{sen}t\operatorname{sen}4t}{t}\right]=0\Rightarrow L\left[\frac{\operatorname{sen}t\operatorname{sen}4t}{t}\right]=\int_s^\infty L\left[\operatorname{sen}t\operatorname{sen}4t\right]du\quad(*)$$

$$\text{sent sen4t} = \frac{1}{2}(\cos 3t - \cos 5t) \Rightarrow L\left[\text{sent sen4t}\right] = \frac{1}{2}\left(\frac{s}{s^2+9} - \frac{s}{s^2+25}\right).$$

$$(*) = \int_s^\infty \frac{1}{2}\left(\frac{u}{u^2+9} - \frac{u}{u^2+25}\right)du = \frac{1}{4}\ln\left(\frac{u^2+9}{u^2+25}\right)\Bigg|_s^\infty = \frac{1}{4}\ln\left(\frac{s^2+25}{s^2+9}\right)$$

pues $\lim_{u\to\infty}\left[\frac{1}{4}\ln\left(\frac{u^2+9}{u^2+25}\right)\right] = 0$

La solución : $\frac{1}{4}\ln\left(\frac{13}{5}\right)$.

5º $L^{-1}\left[\frac{e^{-s}}{(s^2+2s+5)^2}\right] = \begin{cases} F(t-1) & t > 1 \\ 0 & 0 < t < 1 \end{cases}$

Siendo $F(t) = L^{-1}\left[\frac{1}{(s^2+2s+5)^2}\right] = L^{-1}\left[\frac{1}{\left((s+1)^2+4\right)^2}\right] = e^{-t}L^{-1}\left[\frac{1}{(s^2+4)^2}\right]$.

$$f(s) = -\frac{1}{2}\frac{1}{(s^2+4)}; f'(s) = \frac{s}{(s^2+4)^2}$$

$$L^{-1}\left[f'(s)\right] = -tL^{-1}\left[f(s)\right] = \frac{t}{4}\text{sen}2t.$$

$$L^{-1}\left[\frac{1}{(s^2+4)^2}\right] = \int_0^t \frac{u}{4}\text{sen}2u\,du = \frac{1}{4}\left(-\frac{u\cos 2u}{2} + \frac{1}{2}\int \cos 2u\,du\right)\Bigg|_0^t =$$

$$\frac{1}{8}\left(-u\cos 2u + \frac{\text{sen}2u}{2}\right)\Bigg|_0^t = \frac{1}{8}\left(-t\cos 2t + \frac{\text{sen}2t}{2}\right) = \frac{1}{16}\left(-2t\cos 2t + \text{sen}2t\right).$$

La solución $\begin{cases} e^{-(t-1)}\frac{1}{16}\left(-2(t-1)\cos 2(t-1) + \text{sen}2(t-1)\right) & t > 1 \\ 0 & 0 < t < 1 \end{cases}$

6º a) 2l=2, l=1.

$$a_0 = \frac{1}{1}\int_0^2 f(x)dx = \int_0^1 x^2dx + \int_1^2 1dx = \frac{x^3}{3}\Bigg|_0^1 + x\Big|_1^2 = \frac{4}{3}.$$

$$a_k = \frac{1}{1}\int_0^2 f(x)\cos\frac{k\pi x}{1}dx = \int_0^1 x^2\cos k\pi x\,dx + \int_1^2 \cos k\pi x\,dx =$$

$$= x^2 \frac{\operatorname{sen} k\pi x}{k\pi} + \frac{2}{(k\pi)^2} x\cos k\pi x - \frac{2}{(k\pi)^3} \operatorname{sen} k\pi x \Big|_0^1 + \frac{\operatorname{sen} k\pi x}{k\pi}\Big|_1^2 =$$

$$= \frac{2}{(k\pi)^2} \cos k\pi.$$

$$b_k = \frac{1}{1}\int_0^2 f(x)\operatorname{sen}\frac{k\pi x}{1}dx = \int_0^1 x^2 \operatorname{sen} k\pi x dx + \int_1^2 \operatorname{sen} k\pi x dx =$$

$$= -x^2 \frac{\cos k\pi x}{k\pi} + \frac{2}{(k\pi)}\left(\frac{x\operatorname{sen} k\pi x}{k\pi} + \frac{1}{(k\pi)^2}\cos k\pi x\right)\Bigg|_0^1 - \frac{\cos k\pi x}{k\pi}\Bigg|_1^2 =$$

$$= \frac{2}{(k\pi)^3}(\cos k\pi - 1) - \frac{\cos 2k\pi}{k\pi}.$$

$$f(x) = \frac{2}{3} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x - \frac{4}{\pi^3}\sum_{1}^{\infty}\frac{\operatorname{sen}(2n-1)\pi x}{(2n-1)^3} - \frac{1}{\pi}\sum_{1}^{\infty}\frac{\operatorname{sen} n\pi x}{n}$$

x punto de continuidad.

$x_1 = 2k, k = 0. \pm 1, \pm 2, \pm 3....$ puntos de discontinuidad:

$$f(x_1) = \frac{1}{2}.$$

b) Doblamos el periodo para hacerla impar 2l=4, l=2.

$$b_k = \frac{2}{2}\int_0^2 f(x)\operatorname{sen}\frac{k\pi x}{2}dx = \int_0^1 x^2 \operatorname{sen}\frac{k\pi x}{2}dx + \int_1^2 1\operatorname{sen}\frac{k\pi x}{2}dx =$$

$$= -\frac{2x^2}{k\pi}\cos\frac{k\pi x}{2} + \frac{8}{(k\pi)^2}x\operatorname{sen}\frac{k\pi x}{2} + \frac{16}{(k\pi)^3}\cos\frac{k\pi x}{2}\Bigg|_0^1 - \frac{2}{k\pi}\cos\frac{k\pi x}{2}\Bigg|_1^2 =$$

$$= \frac{16}{(k\pi)^3}\left(\cos\frac{k\pi}{2} - 1\right) + \frac{8}{(k\pi)^2}\operatorname{sen}\frac{k\pi}{2} - \frac{2}{k\pi}\cos k\pi.$$

$$f(x) = \frac{16}{\pi^3}\sum_{1}^{\infty}\frac{\left(\cos\frac{n\pi}{2} - 1\right)}{n^3}\operatorname{sen}\frac{n\pi x}{2} + \frac{8}{\pi^2}\sum_{1}^{\infty}(-1)^{n+1}\frac{\operatorname{sen}\frac{(2n-1)}{2}\pi x}{(2n-1)^2} +$$

$$+\frac{2}{\pi}\sum_{1}^{\infty}(-1)^{n+1}\frac{\operatorname{sen}\frac{n\pi x}{2}}{n}$$

x punto de continuidad.

$x_1 = 2k, k = 0. \pm 1, \pm 2, \pm 3....$puntos de discontinuidad: $f(x_1) = 0$