Análisis Matemático

Resolver las siguientes ecuaciones diferenciales:

1º.- $2x^2 y'' - 4y = 2x^3 \ln x - 3\cos(\ln x) + \dfrac{3}{x}$.

2º.- $(x+1)y' + y = -\dfrac{1}{2}(x+1)^4 y^2$.

3º.- $\left(x + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0$.

4º.- Calcular:

$$\int_0^\infty t\,e^{-3t} \cos^2 2t \, dt .$$

5º.- Calcular:

$$L^{-1}\left[\frac{\ln(s+3) - \ln(s+2)}{s}\right].$$

6º.- Desarrollar en serie de coseno de Fourier la función:

$$f(x) = e^{-x} \qquad 0 \le x < 1$$

1º.- Es una ecuación de Euler.

Haciendo $x = e^t \Rightarrow \begin{cases} y_x' = y_t' e^{-t} \\ y_x'' = (y_t'' - y_t')e^{-2t} \end{cases}$

Sustituyendo nos queda:

$$2y_t'' - 2y_t' - 4y = 2te^{3t} - 3\cos t + 3e^{-t} \Rightarrow y_t'' - y_t' - 2y = te^{3t} - \frac{3}{2}\cos t + \frac{3}{2}e^{-t}.$$

$$r^2 - r - 2 = 0 \Rightarrow r = -1, 2 \Rightarrow y_H = Ae^{-t} + Be^{2t}.$$

Hay que calcular tres soluciones particulares:

$$y_{p_1} = (C + Dt)e^{3t} \Rightarrow (y_{p_1})'' - (y_{p_1})' - 2y_{p_1} = te^{3t} \Rightarrow \begin{cases} C = -\frac{5}{16} \\ D = \frac{1}{4} \end{cases} \Rightarrow y_{p_1} = \left(-\frac{5}{16} + \frac{1}{4}t\right)e^{3t}.$$

$$y_{p_2} = E\cos t + F\operatorname{sen} t \Rightarrow (y_{p_2})'' - (y_{p_2})' - 2y_{p_2} = -\frac{3}{2}\cos t \Rightarrow \begin{cases} E = \frac{9}{20} \\ F = \frac{3}{20} \end{cases}$$

$$y_{p_2} = \frac{9}{20}\cos t + \frac{3}{20}\operatorname{sen} t.$$

$$y_{p_3} = Gte^{-t} \Rightarrow (y_{p_3})'' - (y_{p_3})' - 2y_{p_3} = \frac{3}{2}e^{-t} \Rightarrow G = -\frac{1}{2} \Rightarrow y_{p_3} = -\frac{1}{2}te^{-t}.$$

Deshaciendo los cambios:

$$y = Ax^2 + \frac{B}{x} + \left(-\frac{5}{16} + \frac{1}{4}\ln x\right)x^3 + \frac{9}{20}\cos(\ln x) + \frac{3}{20}\operatorname{sen}(\ln x) - \frac{1}{2}\frac{\ln x}{x}.$$

2º.- Es una ecuación de Bernoulli.

$$\frac{y'}{y^2} + \frac{1}{y}\cdot\frac{1}{x+1} = -\frac{1}{2}(x+1)^3, \text{haciendo el cambio } \frac{1}{y} = z \Rightarrow -\frac{y'}{y^2} = z'$$

$$-z' + \frac{z}{x+1} = -\frac{1}{2}(x+1)^3 \text{ lineal.}$$

$$-u'v-uv'+\frac{u\cdot v}{x+1}=-\frac{1}{2}(x+1)^3\Rightarrow v\left(-u'+\frac{u}{x+1}\right)-uv'=-\frac{1}{2}(x+1)^3.$$

$$-u'+\frac{u}{x+1}=0\Rightarrow \ln u=\ln(x+1)\Rightarrow u=x+1.$$

$$-(x+1)v'=-\frac{1}{2}(x+1)^3\Rightarrow v=\frac{1}{2}\int(x+1)^2dx\Rightarrow v=\frac{1}{2}\cdot\frac{(x+1)^3}{3}+C.$$

$$z=\frac{1}{y}=(x+1)\left(\frac{1}{2}\cdot\frac{(x+1)^3}{3}+C\right)\Rightarrow 1=y\cdot(x+1)\left(\frac{1}{2}\cdot\frac{(x+1)^3}{3}+C\right).$$

3º.- Es una diferencial exacta.

$$P=x+e^{\frac{x}{y}}\Rightarrow P_y=-\frac{x}{y^2}e^{\frac{x}{y}};Q=e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\Rightarrow Q_x=\frac{1}{y}e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)-\frac{1}{y}e^{\frac{x}{y}}=-\frac{x}{y^2}e^{\frac{x}{y}}.$$

$$F(x,y)=\int(x+e^{\frac{x}{y}})dx=\frac{x^2}{2}+ye^{\frac{x}{y}}+\alpha(y).$$

$$\frac{\partial F}{\partial y}=e^{\frac{x}{y}}+y\left(\frac{-x}{y^2}\right)e^{\frac{x}{y}}+\alpha'(y)=Q=e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\Rightarrow\alpha'(y)=0\Rightarrow\alpha(y)=cte.$$

Integral general: $\frac{x^2}{2}+ye^{\frac{x}{y}}=K.$

4º.- Aplicando Laplace:

$$\int_0^{\infty}t\,e^{-3t}\cos^2 2t\,dt=\lim_{s\to3}\left(L\left\{t\ \cos^2 2t\right\}\right).$$

$$L\left\{t\ \cos^2 2t\right\}=-f'(s)\ \text{ siendo }\ f(s)=L\left\{\cos^2 2t\right\}.$$

$$\cos^2 2t=\frac{1}{2}(1+\cos4t)\Rightarrow L\left\{\cos^2 2t\right\}=\frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+16}\right).$$

$$f'(s)=\frac{1}{2}\left(-\frac{1}{s^2}+\frac{16-s^2}{(s^2+16)^2}\right)\Rightarrow L\left\{t\ \cos^2 2t\right\}=\frac{1}{2}\left(\frac{1}{s^2}-\frac{16-s^2}{(s^2+16)^2}\right).$$

$$\lim_{s\to3}\left(L\left\{t\ \cos^2 2t\right\}\right)=\frac{1}{2}\left(\frac{1}{9}-\frac{7}{25^2}\right).$$

5º.- $L^{-1}\left[\frac{1}{s}\ln\left(\frac{s+3}{s+2}\right)\right]=\int_0^t F(u)du$, siendo $F(u)=L^{-1}\left[\ln\left(\frac{s+3}{s+2}\right)\right]$.

$$f(s)=\ln\left(\frac{s+3}{s+2}\right)\Rightarrow f'(s)=\frac{1}{s+3}-\frac{1}{s+2}\Rightarrow L^{-1}\left[f'(s)\right]=e^{-3t}-e^{-2t}.$$

$$L^{-1}\left[f'(s)\right]=-tL^{-1}\left[f(s)\right]\Rightarrow L^{-1}\left[f(s)\right]=\frac{e^{-2t}-e^{-3t}}{t}.$$

$$L^{-1}\left[\frac{1}{s}\ln\left(\frac{s+3}{s+2}\right)\right]=\int_0^t \frac{e^{-2u}-e^{-3u}}{u}du.$$

6º.- $2l=2\Rightarrow l=1$.

$$a_0=\frac{2}{1}\int_0^1 e^{-x}dx=2(-e^{-x})\Big|_0^1=2(1-e^{-1}).$$

$a_k=2\int_0^1 e^{-x}\cos k\pi x dx$ cíclica (integrando por partes dos veces)

$$a_k=\frac{2}{1+k^2\pi^2}\left[e^{-x}(-\cos k\pi x+k\pi \operatorname{sen} k\pi x)\right]\Big|_0^1=\frac{2}{1+k^2\pi^2}\left[e^{-1}(-\cos k\pi)+1\right]=$$

$$=\frac{2}{1+k^2\pi^2}\left[e^{-1}(-1)^{k+1}+1\right].$$

$$f(x)=(1-e^{-1})+\sum_{k=1}^{\infty}\frac{2}{1+k^2\pi^2}\left[e^{-1}(-1)^{k+1}+1\right]\cos k\pi x.$$

Siendo x punto de continuidad.