CALCULO

1º Calcular : $\lim_{x\to\infty} x^2 \ln\left(\cos\frac{\pi}{x}\right)$.

2º Hallar el dominio de la función:

$$f(x,y) = \ln(x^2 + y^2 - 4) + \sqrt{1 - \frac{x^2}{16} - \frac{y^2}{4}}\ .$$

3º Sea: $z = \frac{xy}{x-y}$.

Demuestra que se cumple: $x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} = 0$.

4º Calcular: $\int x^2 \ln\left(\frac{1+x}{1-x}\right) dx$.

5º Invertir el orden de integración en:

$$\int_0^1 dy \int_{1-\sqrt{1-y^2}}^{\sqrt{y}} f(x,y)dx + \int_0^1 dy \int_{1+\sqrt{1-y^2}}^{3} f(x,y)dx\ .$$

6º Calcular : $\int_0^4 dx \int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} dy \int_0^{\sqrt{16-x^2-y^2}} dz$. ¿Qué representa?

Solución

1º

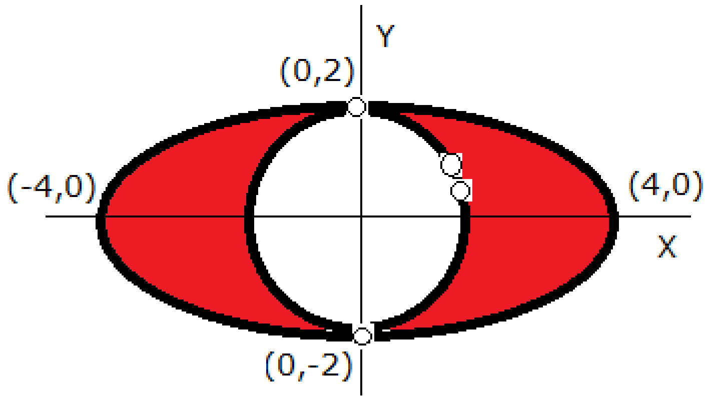
$$\lim_{x\to\infty} x^2 \ln\left(\cos\frac{\pi}{x}\right) = \infty.0 = \lim_{x\to\infty}\frac{\ln\left(\cos\frac{\pi}{x}\right)}{\frac{1}{x^2}} = L\acute{}\,Hôp = \lim_{x\to\infty}\frac{\frac{1}{\cos\frac{\pi}{x}}\left(-sen\frac{\pi}{x}\right)\left(-\frac{\pi}{x^2}\right)}{\frac{-2}{x^3}} = \lim_{x\to\infty}\frac{-\pi x\, sen\frac{\pi}{x}}{2\cos\frac{\pi}{x}} =$$

$$= \lim_{x\to\infty}\frac{-\pi}{2\cos\frac{\pi}{x}}\lim_{x\to\infty} x\, sen\frac{\pi}{x} = \frac{-\pi}{2}\lim_{x\to\infty}\frac{sen\frac{\pi}{x}}{\frac{1}{x}} = L\acute{}\,Hôp = \frac{-\pi}{2}\lim_{x\to\infty}\frac{\cos\frac{\pi}{x}\left(-\frac{\pi}{x^2}\right)}{-\frac{1}{x^2}} = \frac{-\pi^2}{2}.$$

2º

$$D(f) = \left\{(x,y)\,/\,x^2+y^2-4>0 \;\; y\ (\wedge)\ 1-\frac{x^2}{16}-\frac{y^2}{4}\geq 0\right\} =$$

$$= \left\{(x,y)\,/\,x^2+y^2>4 \;\; y\ (\wedge)\ 1\geq\frac{x^2}{16}+\frac{y^2}{4}\right\}$$



3º

$$z_x = \frac{-y^2}{(x-y)^2} \Rightarrow \begin{cases} z_{xx} = \dfrac{2y^2}{(x-y)^3} \\ z_{xy} = \dfrac{-2xy}{(x-y)^3} \end{cases}$$

$$z_y = \frac{x^2}{(x-y)^2} \Rightarrow z_{yy} = \frac{2x^2}{(x-y)^3}$$

$$x^2 z_{xx} + 2xy\, z_{xy} + y^2 z_{yy} = \frac{2x^2y^2}{(x-y)^3} + 2xy\left(\frac{-2xy}{(x-y)^3}\right) + \frac{2x^2y^2}{(x-y)^3} = 0.$$

4º

$$\int x^2 \ln\left(\frac{1+x}{1-x}\right)dx = \frac{x^3}{3}\ln\left(\frac{1+x}{1-x}\right) - \int \frac{x^3}{3}.\frac{2}{(1-x^2)}dx = \frac{x^3}{3}\ln\left(\frac{1+x}{1-x}\right) + \frac{2}{3}\int \frac{x^3}{(x^2-1)}dx \quad \textbf{(*)}$$

$$\frac{x^3}{(x^2-1)} = x + \frac{x}{(x^2-1)}$$

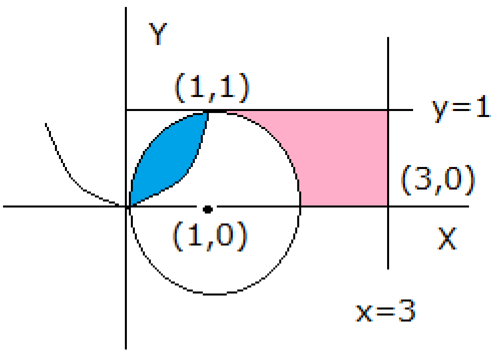
$$\int\left(x + \frac{x}{(x^2-1)}\right)dx = \frac{x^2}{2} + \frac{1}{2}\ln(x^2-1)$$

$$\textbf{(*)} = \frac{x^3}{3}\ln\left(\frac{1+x}{1-x}\right) + \frac{1}{3}\left(x^2 + \ln(x^2-1)\right) + cte.$$

5º

$$\int_0^1 dy \int_{1-\sqrt{1-y^2}}^{\sqrt{y}} f(x,y)dx + \int_0^1 dy \int_{1+\sqrt{1-y^2}}^{3} f(x,y)dx =$$

$$\int_0^1 dx \int_{x^2}^{\sqrt{2x-x^2}} f(x,y)dy + \int_1^2 dx \int_{\sqrt{2x-x^2}}^{1} f(x,y)dy + \int_2^3 dx \int_0^1 f(x,y)dy.$$



6º

$$\int_0^4 dx \int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} dy \int_0^{\sqrt{16-x^2-y^2}} dz = \int_0^4 dx \int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}} \sqrt{16-x^2-y^2}\, dy \quad \text{en polares}$$

$$\iint_D \sqrt{16-r^2}\; r\, d\alpha\, dr \quad ;\text{D es el interior de } (x-2)^2+y^2=4;\ r^2-4r\cos\alpha=0.$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \int_0^{4\cos\alpha} \sqrt{16-r^2}\; r\, dr \quad \mathbf{(*)} \quad ; \quad \int_0^{4\cos\alpha} \sqrt{16-r^2}\; r\, dr = -\left.\frac{\left(\sqrt{16-r^2}\right)^3}{3}\right|_0^{4\cos\alpha} =$$

$$= -\frac{1}{3}\left[\left(\sqrt{16(1-\cos^2\alpha)}\right)^3 - 4^3\right] = \frac{4^3}{3}\left[1-\text{sen}^3\alpha\right]$$

$$\mathbf{(*)} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4^3}{3}\left[1-\text{sen}^3\alpha\right] d\alpha = \frac{4^3}{3}\pi - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \text{sen}^3\alpha\, d\alpha;$$

$$\int \text{sen}^3\alpha\, d\alpha = \int -(1-t^2)dt = \frac{t^3}{3} - t \quad (\text{siendo } t=\cos\alpha).$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \text{sen}^3\alpha\, d\alpha = \text{por ser función impar sen}^3\alpha = 2\int_0^{\frac{\pi}{2}} \text{sen}^3\alpha\, d\alpha =$$

$$2\left[\frac{\cos^3\alpha}{3} - \cos\alpha\right]_0^{\frac{\pi}{2}} = 2\left[-\frac{1}{3}+1\right] = \frac{4}{3}$$

$$\mathbf{(*)} = \frac{4^3}{3}\pi - \frac{4}{3} = \frac{4}{3}\left(4^2\pi - 1\right) \text{ unidades de volumen.}$$

El valor calculado representa el volumen de un cuerpo cilíndrico, de sección la circunferencia $(x-2)^2+y^2=4$ limitado por abajo por $z=0$ y por arriba por la esfera $x^2+y^2+z^2=16$.