CÁLCULO

1º.- Sea $f(x) = \begin{cases} \operatorname{arctag}\left(\dfrac{1+x}{2-x}\right) & x \neq 2 \\ \dfrac{\pi}{2} & x = 2 \end{cases}$

a) Estudiar la continuidad, clasificando los posibles puntos de discontinuidad.

b) Calcular $f'(2^+)$ y $f'(2^-)$.

2º.- Hallar: $\displaystyle\int \frac{x+1}{\sqrt{x^2+x+1}}\,dx.$

3º.- Sea $z = \ln(x^2+y^2) + \operatorname{arctg}\left(\dfrac{y}{x}\right).$

a) Hallar su dominio.

b) Calcular $\left(\dfrac{\partial z}{\partial x}\right)_{(1,0)} \left(\dfrac{\partial z}{\partial y}\right)_{(1,0)}$ ¿En qué dirección la derivada en el punto (1,0) es máxima? Hallarla.

c) Calcular $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}.$

4º.- Invertir el orden de integración en:

$$\int_{-1}^{0} dy \int_{-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx + \int_{0}^{1} dy \int_{1-\sqrt{1-y^2}}^{y^2} f(x,y)\,dx + \int_{0}^{1} dy \int_{1+\sqrt{1-y^2}}^{2} f(x,y)\,dx.$$

5º.- Calcular: $\displaystyle\iiint_V \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz.$

V es el interior de la esfera $x^2+y^2+z^2 = 2y.$

1º.- a) D(f) = R. El único punto conflictivo es x = 2, pues en todos los demás el límite coincide con el valor de la función en el punto luego es continua.

x = 2.

$\lim_{x\to 2^+} \operatorname{arctg}\left(\frac{1+x}{2-x}\right) = -\frac{\pi}{2}; \quad \lim_{x\to 2^-} \operatorname{arctg}\left(\frac{1+x}{2-x}\right) = \frac{\pi}{2}$, como no coinciden no existe límite en el punto x = 2 $\Rightarrow$ discontinuidad inevitable de 1ª especie con salto finito.

$$b)\ f'(2^+) = \lim_{h\to 0}\left(\frac{f(2+h)-f(2)}{h}\right) = \lim_{h\to 0}\left(\frac{\operatorname{arctg}\left(\frac{1+(2+h)}{2-(2+h)}\right)-\frac{\pi}{2}}{h}\right) \to \frac{-\pi}{0} \to -\infty.$$

$$f'(2^-) = \lim_{h\to 0}\left(\frac{f(2-h)-f(2)}{-h}\right) = \lim_{h\to 0}\left(\frac{\operatorname{arctg}\left(\frac{1+(2-h)}{2-(2-h)}\right)-\frac{\pi}{2}}{-h}\right) = \text{L´Hôp} = \frac{1}{3}.$$

2º.- Completando cuadrados: $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left[\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right] =$

$$= \frac{3}{4}\left[\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right] \Rightarrow \sqrt{x^2+x+1} = \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}.$$

Haciendo el cambio $\frac{2x+1}{\sqrt{3}} = t,\ x = \frac{\sqrt{3}\,t-1}{2},\ dx = \frac{\sqrt{3}}{2}dt.$

$$\int \frac{\frac{\sqrt{3}\,t-1}{2}+1}{\frac{\sqrt{3}}{2}\sqrt{t^2+1}}\frac{\sqrt{3}}{2}dt = \frac{1}{2}\int\frac{\sqrt{3}\,t+1}{\sqrt{t^2+1}} = \frac{1}{2}\left[\sqrt{3}\sqrt{t^2+1}+\ln\left(t+\sqrt{t^2+1}\right)\right]+\text{Cte}.$$

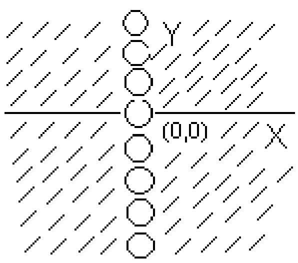
Deshaciendo el cambio $\frac{1}{2}\left[\sqrt{3}\sqrt{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}+\ln\left(\frac{2x+1}{\sqrt{3}}+\sqrt{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\right)\right]+\text{Cte} =$

$$= \frac{1}{2}\sqrt{4x^2+4x+4}+\frac{1}{2}\ln\left[\frac{1}{\sqrt{3}}\left(2x+1+\sqrt{4x^2+4x+4}\right)\right]+\text{Cte} =$$

$$= \sqrt{x^2+x+1}+\frac{1}{2}\ln\left(2x+1+2\sqrt{x^2+x+1}\right)+K = \sqrt{x^2+x+1}+\frac{1}{2}\ln\left[\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right]+K'.$$

3º.- a) $D\left[\ln(x^2+y^2)\right] = R^2-(0,0);\quad D\left[\operatorname{arctg}\left(\frac{y}{x}\right)\right] = R^2-\{(0,y)/y\in R\}.$

La intersección nos da:



$$\text{b)}\left(\frac{\partial z}{\partial x}\right)_{(1,0)}=\left(\frac{2x}{x^2+y^2}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right)\right)_{(1,0)}=\left(\frac{2x-y}{x^2+y^2}\right)_{(1,0)}=2.$$

$$\left(\frac{\partial z}{\partial y}\right)_{(1,0)}=\left(\frac{2y}{x^2+y^2}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)\right)_{(1,0)}=\left(\frac{2y+x}{x^2+y^2}\right)_{(1,0)}=1.$$

La derivada en el punto (1,0) es máxima en la dirección del vector gradiente

$\overrightarrow{grad}(z)_{(1,0)}=2\vec{i}+\vec{j}.$

Su valor es el módulo del vector gradiente $\sqrt{5}$.

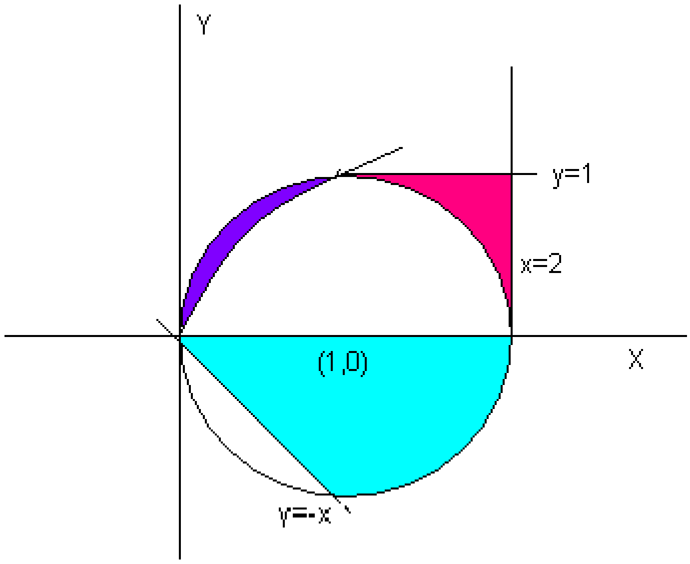
$$\text{c)}\ \frac{\partial z}{\partial x}=\frac{2x}{x^2+y^2}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)=\frac{2x}{x^2+y^2}+\frac{(-y)}{x^2+y^2}=\frac{2x-y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{2(x^2+y^2)-2x.(2x-y)}{(x^2+y^2)^2}=\frac{2y^2-2x^2+2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial z}{\partial y}=\frac{2y}{x^2+y^2}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)=\frac{2y}{x^2+y^2}+\frac{x}{x^2+y^2}=\frac{2y+x}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{2(x^2+y^2)-2y.(2y+x)}{(x^2+y^2)^2}=\frac{2x^2-2y^2-2xy}{(x^2+y^2)^2}\Rightarrow\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0.$$

4º.-



$x^2 + y^2 - 2x = (x-1)^2 + y^2 = 1$, circunferencia C(1,0) y radio 1.

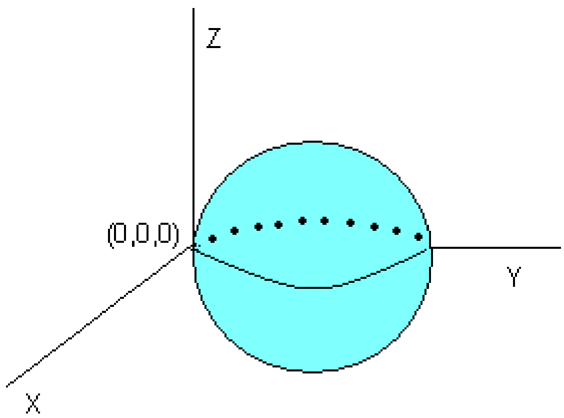
$y^2 = x$ parábola vértice (0,0) y de eje el OX.

$y = -x$, recta.

Invirtiendo el orden de integración:

$$\int_0^1 dx \int_{-x}^{0} f(x,y)\,dy + \int_0^1 dx \int_{\sqrt{x}}^{\sqrt{2x-x^2}} f(x,y)\,dy + \int_1^2 dx \int_{-\sqrt{2x-x^2}}^{0} f(x,y)\,dy + \int_1^2 dx \int_{\sqrt{2x-x^2}}^{1} f(x,y)\,dy.$$

5º.-



$x^2+y^2+z^2=2y \Rightarrow x^2+(y-1)^2+z^2=1$, esfera de $C(0,1,0)$ y radio 1.

Utilizando las coordenadas esféricas:

$x^2+y^2+z^2=r^2$; $2y=2r\,\text{sen}\alpha\cos\beta$; Jacobiano $=r^2\cos\beta$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\beta \int_0^{\pi} d\alpha \int_0^{2\text{sen}\alpha\cos\beta} \sqrt{r^2}\; r^2\cos\beta\, dr.$$

$$\int_0^{2\text{sen}\alpha\cos\beta} r^3 dr = \left.\frac{r^4}{4}\right|_0^{2\text{sen}\alpha\cos\beta} = 4\text{sen}^4\alpha\cos^4\beta.$$

$$\int \cos^5\beta\, d\beta = \begin{cases}\text{sen}\beta = t\\ \cos\beta\, d\beta = dt\end{cases} \Rightarrow \int (1-t^2)^2 dt = \frac{t^5}{5} - 2\frac{t^3}{3} + t = \frac{\text{sen}^5\beta}{5} - 2\frac{\text{sen}^3\beta}{3} + \text{sen}\beta.$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5\beta\, d\beta = \left.\frac{\text{sen}^5\beta}{5} - 2\frac{\text{sen}^3\beta}{3} + \text{sen}\beta\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\left(\frac{1}{5}-\frac{2}{3}+1\right) = \frac{16}{15}.$$

$$\int \text{sen}^4\alpha\, d\alpha = \int \left(\frac{1-\cos 2\alpha}{2}\right)^2 d\alpha = \frac{1}{4}\int \left(1+\cos^2 2\alpha + 2\cos 2\alpha\right) d\alpha = \frac{1}{4}\int \left(1+\frac{1+\cos 4\alpha}{2} + 2\cos 2\alpha\right) d\alpha =$$

$$= \frac{1}{4}\left[\alpha + \frac{1}{2}\left(\alpha + \frac{\text{sen}4\alpha}{4}\right) + \text{sen}2\alpha\right] = \frac{1}{4}\left(\frac{3}{2}\alpha + \frac{1}{8}\text{sen}4\alpha + \text{sen}2\alpha\right).$$

$$\int_0^{\pi} \text{sen}^4\alpha\, d\alpha = \left.\frac{1}{4}\left(\frac{3}{2}\alpha + \frac{1}{8}\text{sen}4\alpha + \text{sen}2\alpha\right)\right|_0^{\pi} = \frac{3}{8}\pi.$$

Total: $4.\frac{16}{15}.\frac{3}{8}\pi = \frac{8}{5}\pi.$