CÁLCULO

1º.- Sea La función:

$$f(x)=\begin{cases} \sin x & x>0 \\ 0 & x=0 \\ x+x^2 & x<0 \end{cases}$$

Calcular $f''(x)$.

2º.- a) Sea $u=\ln\left(\dfrac{1}{\sqrt{ax^2+by^2}}\right)$.

Demostrar que se cumple: $b\,u_{xx} + a\,u_{yy} = 0$.

b) Dado $v=\left(\sqrt{\dfrac{x^2+y^2}{x-y}}\right)^{\frac{x}{y}}$.Calcular v_x .

3º.- Cambiar el orden de integración y dibujar el recinto en:

$$\int_0^3 dy \int_{-\sqrt{9-y^2}}^{-\frac{2}{3}\sqrt{9-y^2}} f(x,y)\,dx + \int_0^{\frac{3\sqrt{3}}{2}} dy \int_{\frac{2}{3\sqrt{3}}y}^{\frac{2}{3}\sqrt{9-y^2}} f(x,y)\,dx .$$

4º.- Calcular: $\displaystyle\int \frac{dx}{e^{6x}+e^{3x}+1}$.

5º.- Hallar: $\displaystyle\iiint_V xyz\,dx\,dy\,dz$.

V es el recinto limitado por las superficies:

$x^2+y^2+z^2=1, x=0, y=0, z=0.$

Solución

1º.-

$$f'(0^+)=\lim_{h\to 0}\frac{\text{sen}(0+h)-0}{h}=1;\ f'(0^-)=\lim_{h\to 0}\frac{(0-h)+(0-h)^2-0}{-h}=1.$$

$$f'(x)=\begin{cases}\cos x & x>0\\ 1 & x=0\\ 1+2x & x<0\end{cases}$$

$$\left.\begin{array}{l} f''(0^+)=\lim_{h\to 0}\frac{\cos(0+h)-1}{h}=\text{L'Hôpital}=\lim_{h\to 0}(-\text{sen}h)=0.\\ f''(0^-)=\lim_{h\to 0}\frac{1+2(0-h)-1}{-h}=2\end{array}\right\}\Rightarrow \text{No existe } f''(0).$$

$$f''(x)=\begin{cases}-\text{sen}x & x>0\\ 2 & x<0\end{cases}$$

2º.-

a) $u=-\frac{1}{2}\ln(ax^2+by^2);u_x=-\frac{1}{2}\frac{2ax}{ax^2+by^2}=\frac{-ax}{ax^2+by^2};u_{xx}=-a\left[\frac{ax^2+by^2-2ax^2}{\left(ax^2+by^2\right)^2}\right]=$

$=a\left[\frac{ax^2-by^2}{\left(ax^2+by^2\right)^2}\right].$

$u_y=-\frac{1}{2}\frac{2by}{ax^2+by^2}=\frac{-by}{ax^2+by^2};u_{yy}=-b\left[\frac{ax^2+by^2-2by^2}{\left(ax^2+by^2\right)^2}\right]=b\left[\frac{by^2-ax^2}{\left(ax^2+by^2\right)^2}\right].$

$bu_{xx}+au_{yy}=ba\left[\frac{ax^2-by^2}{\left(ax^2+by^2\right)^2}\right]+ab\left[\frac{by^2-ax^2}{\left(ax^2+by^2\right)^2}\right]=0.$

b) $v=\left(\frac{x^2+y^2}{x-y}\right)^{\frac{x}{2y}}\Rightarrow v=m^p\begin{cases}m=\frac{x^2+y^2}{x-y}\\ p=\frac{x}{2y}\end{cases}\Rightarrow\frac{\partial v}{\partial x}=\frac{\partial v}{\partial m}.\frac{\partial m}{\partial x}+\frac{\partial v}{\partial p}.\frac{\partial p}{\partial x}=$

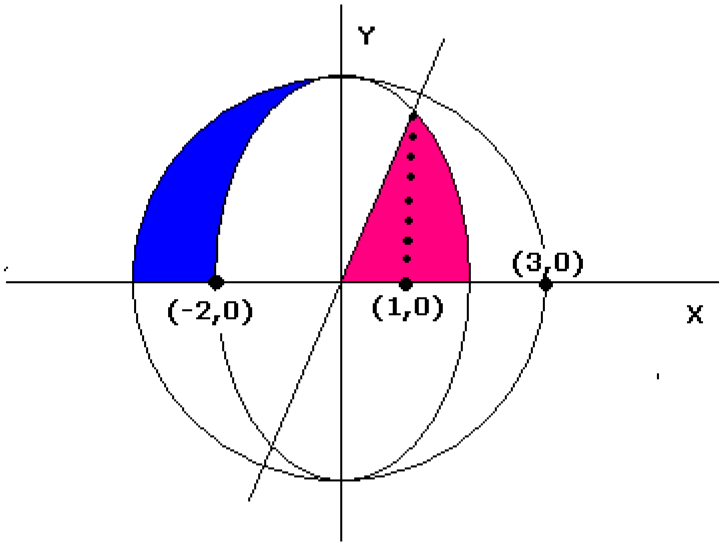
$=pm^{p-1}.m_x+m^p\ln(m).p_x$

Sustituyendo: $\frac{x}{2y}\left(\frac{x^2+y^2}{x-y}\right)^{\frac{x}{2y}-1}\left(\frac{x^2-y^2-2xy}{(x-y)^2}\right)+\left(\frac{x^2+y^2}{x-y}\right)^{\frac{x}{2y}}\ln\left(\frac{x^2+y^2}{x-y}\right)\frac{1}{2y}.$

Nota.- Hacerlo tomando neperianos y comprobar el resultado.

3º.- Las curvas que encierran el dominio son:

$$x^2 = \frac{4}{9}(9 - y^2);9x^2 + 4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1 \text{ (elipse)}.$$
$$x^2 = (9 - y^2);x^2 + y^2 = 9 \text{ (circunferencia)}.$$
$$x = \frac{2}{3\sqrt{3}}y \text{ (recta)}.$$



$$\int_{-3}^{-2} dx \int_{0}^{\sqrt{9-x^2}} f(x,y)\,dy + \int_{-2}^{0} dx \int_{\frac{3}{2}\sqrt{4-x^2}}^{\sqrt{9-x^2}} f(x,y)\,dy + \int_{0}^{1} dx \int_{0}^{\frac{3\sqrt{3}}{2}x} f(x,y)\,dy + \int_{1}^{2} dx \int_{0}^{\frac{3}{2}\sqrt{4-x^2}} f(x,y)\,dy.$$

4º.-

$$\left.\begin{array}{l} e^{3x} = t \\ 3e^{3x}dx = dt \end{array}\right\} \Rightarrow \int \frac{dx}{e^{6x} + e^{3x} + 1} = \frac{1}{3}\int \frac{dt}{t(t^2 + t + 1)}.$$

$$\frac{1}{t(t^2+t+1)}=\frac{A}{t}+\frac{Bt+C}{(t^2+t+1)}\Rightarrow\begin{cases}A=1\\B=-1\\C=-1\end{cases}$$

$$\frac{1}{3}\left[\ln t-\int\frac{t+1}{(t^2+t+1)}dt\right];(t^2+t+1)=\left(t+\frac{1}{2}\right)^2+\frac{3}{4}=\frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}(t+\frac{1}{2})\right)^2+1\right]=$$

$$\frac{3}{4}\left[\left(\frac{2t+1}{\sqrt{3}}\right)^2+1\right]\text{haciendo el cambio}\begin{cases}\dfrac{2t+1}{\sqrt{3}}=u\\dt=\dfrac{\sqrt{3}}{2}du\end{cases}$$

$$\int\frac{t+1}{(t^2+t+1)}dt=\int\frac{\frac{\sqrt{3}u-1}{2}+1}{\frac{3}{4}(u^2+1)}\frac{\sqrt{3}}{2}du=\frac{\sqrt{3}}{3}\int\frac{\sqrt{3}u+1}{(u^2+1)}du=\frac{1}{2}\ln(u^2+1)+\frac{\sqrt{3}}{3}\operatorname{arctg}u+cte.$$

$$\frac{1}{2}\ln\left(\frac{4}{3}(t^2+t+1)\right)+\frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2t+1}{\sqrt{3}}\right)+cte=\frac{1}{2}\ln\left(t^2+t+1\right)+\frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2t+1}{\sqrt{3}}\right)+K.$$

$$\frac{1}{3}\left[\ln t-\int\frac{t+1}{(t^2+t+1)}dt\right]=\frac{1}{3}\left[\ln t-\frac{1}{2}\ln\left(t^2+t+1\right)-\frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2t+1}{\sqrt{3}}\right)\right]+C.$$

Donde $t=e^{3x}$.Sustituyendo:

$$x-\frac{1}{6}\ln\left(e^{6x}+e^{3x}+1\right)-\frac{\sqrt{3}}{9}\operatorname{arctg}\left(\frac{2e^{3x}+1}{\sqrt{3}}\right)+Cte.$$

5º.- Se puede resolver en: cartesianas, cilíndricas y esféricas.

$$\int_0^1 xdx\int_0^{\sqrt{1-x^2}}ydy\int_0^{\sqrt{1-x^2-y^2}}zdz=\frac{1}{2}\int_0^1 xdx\int_0^{\sqrt{1-x^2}}y\left(1-x^2-y^2\right)dy.$$

Pasando a polares:

$$\frac{1}{2}\int_0^{\frac{\pi}{2}}\operatorname{sen}\alpha\cos\alpha\,d\alpha\int_0^1 r^3(1-r^2)dr.$$

$$\int_0^1 r^3(1-r^2)dr=\left.\frac{r^4}{4}-\frac{r^6}{6}\right|_0^1=\frac{1}{12}.$$

$$\int_0^{\frac{\pi}{2}}\operatorname{sen}\alpha\cos\alpha\,d\alpha=\left.\frac{\operatorname{sen}^2\alpha}{2}\right|_0^{\frac{\pi}{2}}=\frac{1}{2}.$$

Total: $\frac{1}{2}.\frac{1}{12}.\frac{1}{2}=\frac{1}{48}.$