CÁLCULO

1º.- Sea la función:

$$f(x) = \frac{\ln x - \operatorname{sen}(x-1)}{x - x^x}.$$

- **a)** Hallar su dominio.
- **b)** Calcular: $\lim_{x \to 1} f(x)$.
- **c)** ¿Es continua la función en x=1? Caso de discontinuidad, clasificar el punto.

2º.- Dada la función:

$$z(x,y) = \ln(x^2 + y^2) + \operatorname{arctg}\frac{y}{x}.$$

Calcular: $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}$.

3º.- Hallar la longitud del arco de curva de la función:

$$f(x) = \ln(\cos x) \qquad \text{de } x = 0 \text{ a } x = \frac{\pi}{3}.$$

4º.- Invertir el orden de integración en:

$$\int_{-1}^{0} dx \int_{x^2}^{-x} f(x,y)\,dy + \int_{0}^{1} dx \int_{x^2}^{\sqrt{2x-x^2}} f(x,y)\,dy + \int_{1}^{2} dx \int_{\sqrt{2x-x^2}}^{x^2} f(x,y)\,dy.$$

5º.- Hallar el volumen de la región limitada por:

$$z = x^2 + y^2 \qquad y \qquad z = 2x$$

Solución

1º.- a) $D(\ln x)=(0,\infty); D(\operatorname{sen}(x-1))=R; D(x^x)=(0,\infty)$ y el denominador se hace cero para x=1. Luego $D(f(x))=(0,1)\cup(1,\infty)$.

b) $\lim\limits_{x\to 1} f(x)=\frac{0}{0} L'Hôpital=\lim\limits_{x\to 1}\frac{\frac{1}{x}-\cos(x-1)}{1-x^x(\ln x+1)}=\frac{0}{0}.$

$$y=x^x \Rightarrow \ln y = x\ln x \Rightarrow \frac{y'}{y}=\ln x + x.\frac{1}{x}\Rightarrow y'=x^x(\ln x+1).$$

Operando: $\lim\limits_{x\to 1}\frac{1}{x}.\frac{1-x\cos(x-1)}{1-x^x(\ln x+1)}=\lim\limits_{x\to 1}\frac{1}{x}.\lim\limits_{x\to 1}\frac{1-x\cos(x-1)}{1-x^x(\ln x+1)}=\lim\limits_{x\to 1}\frac{1-x\cos(x-1)}{1-x^x(\ln x+1)}=\frac{0}{0}.$

Volviendo aplicar L´Hôpital; $\lim\limits_{x\to 1}\frac{-\cos(x-1)+x\operatorname{sen}(x-1)}{-x^x(\ln x+1)^2-\frac{x^x}{x}}=\frac{-1}{-2}=\frac{1}{2}.$

c) La función tiene límite en x=1 pero no está definida, luego es un punto de "**discontinuidad evitable**".

2º.- $\frac{\partial z}{\partial x}=\frac{2x}{x^2+y^2}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)=\frac{2x}{x^2+y^2}+\frac{(-y)}{x^2+y^2}=\frac{2x-y}{x^2+y^2}$

$$\frac{\partial^2 z}{\partial x^2}=\frac{2(x^2+y^2)-2x.(2x-y)}{(x^2+y^2)^2}=\frac{2y^2-2x^2+2xy}{(x^2+y^2)^2}.$$

$$\frac{\partial z}{\partial y}=\frac{2y}{x^2+y^2}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)=\frac{2y}{x^2+y^2}+\frac{x}{x^2+y^2}=\frac{2y+x}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{2(x^2+y^2)-2y.(2y+x)}{(x^2+y^2)^2}=\frac{2x^2-2y^2-2xy}{(x^2+y^2)^2}\Rightarrow\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0.$$

3º.- $I=\int_0^{\frac{\pi}{3}}\sqrt{1+(f'(x))^2}\,dx=\int_0^{\frac{\pi}{3}}\sqrt{1+\left(\frac{-\operatorname{sen}x}{\cos x}\right)^2}\,dx=\int_0^{\frac{\pi}{3}}\sqrt{\frac{1}{\cos^2 x}}\,dx=\int_0^{\frac{\pi}{3}}\frac{1}{\cos x}\,dx.$

Aplicando el cambio: $\operatorname{tg}\frac{x}{2} = t \Rightarrow \begin{cases} \cos x = \frac{1-t^2}{1+t^2} \to \frac{1}{\cos x} = \frac{1+t^2}{1-t^2} \\ dx = \frac{2dt}{1+t^2} \end{cases}$

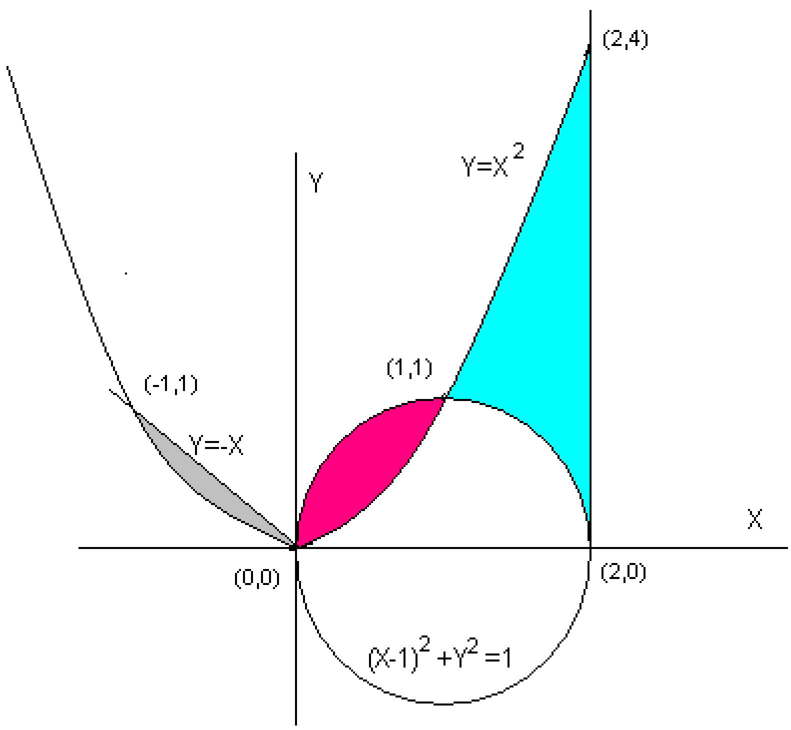
Resolviendo la indefinida: $\int \frac{1+t^2}{1-t^2} \cdot \frac{2dt}{1+t^2} = \int \frac{2dt}{1-t^2}$ descomponiendo

$$\frac{2}{1-t^2} = \frac{A}{1-t} + \frac{B}{1+t} \Rightarrow \begin{cases} A = 1 \\ B = 1 \end{cases} \Rightarrow \int \frac{2dt}{1-t^2} = -\ln(1-t) + \ln(1+t) = \ln\frac{1+t}{1-t}.$$

$$I = \ln\left(\frac{1+\operatorname{tg}\frac{x}{2}}{1-\operatorname{tg}\frac{x}{2}}\right)\Bigg|_0^{\frac{\pi}{3}} = \ln\left(\frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}\right) = \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) = \ln\left(\frac{(\sqrt{3}+1)^2}{3-1}\right) =$$

$$= \ln\left(\frac{3+1+2\sqrt{3}}{2}\right) = \ln(2+\sqrt{3}) \text{ unidades de longitud}$$

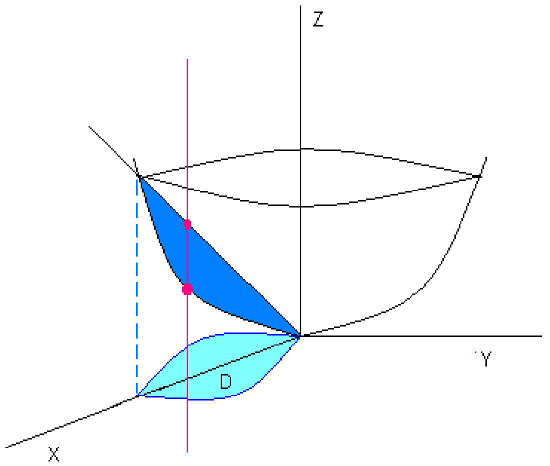
4º.-

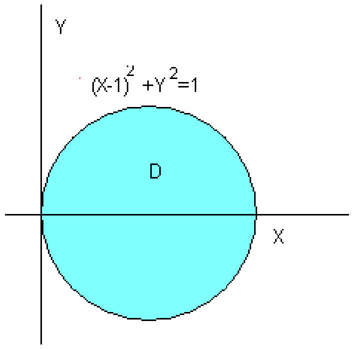


Las curvas que encierran el dominio son: $\begin{cases} y=-x \\ y=x^2 \\ y=\sqrt{2x-x^2} \Rightarrow (x-1)^2+y^2=1 \\ x=2 \end{cases}$

$$\int_0^1 dy \int_{-\sqrt{y}}^{-y} f(x,y)dx + \int_0^1 dy \int_{1-\sqrt{1-y^2}}^{\sqrt{y}} f(x,y)dx + \int_0^1 dy \int_{1+\sqrt{1-y^2}}^{2} f(x,y)dx + \int_1^4 dy \int_{\sqrt{y}}^{2} f(x,y)dx \; .$$

5º.- El volumen es el del paraboloide cortado por el plano z=2x.





$$z=2x=x^2+y^2 \Rightarrow x^2+y^2-2x=0 \Rightarrow (x-1)^2+y^2=1.$$

$$V=\iint_D dx\,dy \int_{x^2+y^2}^{2x} dz = \iint_D (2x-x^2-y^2)dx\,dy \qquad \text{pasando a polares}$$

$$V=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \int_0^{2\cos\alpha} (2r\cos\alpha - r^2)r\,dr.$$

$$\int_0^{2\cos\alpha} (2r^2\cos\alpha - r^3)dr = \left.\frac{2r^3}{3}\cos\alpha - \frac{r^4}{4}\right|_0^{2\cos\alpha} = \left(\frac{16}{3}-\frac{16}{4}\right)\cos^4\alpha =$$

$$=16\left(\frac{1}{3}-\frac{1}{4}\right)\cos^4\alpha = \frac{4}{3}\cos^4\alpha \qquad (*)$$

Haciendo la indefinida:

$$\int \cos^4 \alpha\, d\alpha = \int \left(\frac{1+\cos 2\alpha}{2}\right)^2 d\alpha = \frac{1}{4}\int (1+\cos^2 2\alpha + 2\cos 2\alpha)\, d\alpha =$$

$$\frac{1}{4}\int \left[1+\left(\frac{1+\cos 4\alpha}{2}\right)+2\cos 2\alpha\right] d\alpha = \frac{1}{4}\int \left(\frac{3}{2}+\frac{\cos 4\alpha}{2}+2\cos 2\alpha\right) d\alpha =$$

$$= \frac{1}{4}\left(\frac{3\alpha}{2}+\frac{\operatorname{sen} 4\alpha}{8}+\operatorname{sen} 2\alpha\right).$$

Volviendo a (*)

$$V = \frac{4}{3}.\frac{1}{4}\left(\frac{3\alpha}{2}+\frac{\operatorname{sen} 4\alpha}{8}+\operatorname{sen} 2\alpha\right)\Bigg|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{3}.\frac{3}{2}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{2} \text{ unidades de volumen.}$$