1º.- Operar en forma polar, dando el resultado en forma binómica:

$$\frac{\left(-\sqrt{8} - \sqrt{8}\,i\right)^2 \left(\sqrt{2} - \sqrt{2}\,i\right)^6 \left(-5\sqrt{3} + 5i\right)^5}{\left(2 - 2\sqrt{3}\,i\right)^4}.$$

2º.- Dada la función: $f(x) = \begin{cases} \dfrac{2 + e^{\frac{3}{x}}}{1 + e^{\frac{3}{x}}} & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases}$

a) Estudiar el tipo de discontinuidad que presenta en x=0.
b) ¿Cómo debe definirse la función para que sea continua a la izquierda de x=0?
c) Hallar la función derivada.

3º.- Calcular el valor que toma la expresión $u = x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y}$ siendo:

$$z = \operatorname{sen}\left(\frac{2x + y}{2x - y}\right).$$

4º.- Calcular $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ en el punto $P(\frac{\pi}{4},\frac{\pi}{4})$ de la función:

$$z = \ln(x^x + y^y) + \sqrt{\operatorname{sen}(x + y)}.$$

5º.- Calcular $\iint_D (x + y)^2\,dx\,dy$, siendo D el recinto definido por:

$$\begin{cases} x^2 + y^2 - 2y \geq 0 \\ x^2 + y^2 - 2x \leq 0 \\ y \geq 0 \end{cases}$$

6º.- Calcular $\iiint_V (z^2 + 1)\,dx\,dy\,dz$, siendo V el dominio limitado por:

$$\begin{cases} \dfrac{x^2}{9} + \dfrac{y^2}{4} = 2z \\ z = 4 \end{cases}$$

7º.- Hallar: $\displaystyle\int \frac{xe^{3x}}{\sqrt{1 + e^{3x}}}\,dx$

Solución

1º. $\left(-\sqrt{8} - \sqrt{8}\,i\right)^2 = \left(4_{225^\circ}\right)^2 = 4^2{}_{450^\circ}$;

$\left(\sqrt{2} - \sqrt{2}\,i\right)^6 = \left(2_{315^\circ}\right)^6 = \left(2_{-45^\circ}\right)^6 = 2^6{}_{-270^\circ}$;

$\left(-5\sqrt{3} + 5i\right)^5 = \left(10_{150^\circ}\right)^5 = 10^5{}_{750^\circ}$;

$\left(2 - 2\sqrt{3}\,i\right)^4 = \left(4_{300^\circ}\right)^4 = 4^4{}_{1200^\circ}$;

$$\frac{\left(4^2{}_{450^\circ}\right)\left(2^6{}_{-270^\circ}\right)\left(10^5{}_{750^\circ}\right)}{4^4{}_{1200^\circ}} = \left(\frac{4^2.2^6.10^5}{4^4}\right)_{(450^\circ-270^\circ+750^\circ-1200^\circ)} = 2^7.5^5{}_{-270^\circ} = 2^7.5^5{}_{90}$$

En forma binómica: $2^7.5^5\,i$.

2º. a) $f(0)=0$, los límites laterales:

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0}\left(\frac{2+e^{\frac{3}{h}}}{1+e^{\frac{3}{h}}}\right) = \frac{\infty}{\infty} = \lim_{h\to 0}\left(\frac{2e^{-\frac{3}{h}}+1}{e^{-\frac{3}{h}}+1}\right) = 1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0}\left(\frac{2+e^{\frac{3}{-h}}}{1+e^{\frac{3}{-h}}}\right) = 2.$$

En $x=0$, presenta una discontinuidad inevitable de 1ª especie con salto finito.

b) $f(0)=2$.

c) Al no ser continua en $x=0$, no tiene derivada en dicho punto.

$$f'(x) = \frac{-\frac{3}{x^2}e^{\frac{3}{x}}\left(1+e^{\frac{3}{x}}\right) + \frac{3}{x^2}e^{\frac{3}{x}}\left(2+e^{\frac{3}{x}}\right)}{\left(1+e^{\frac{3}{x}}\right)^2} = \frac{3e^{\frac{3}{x}}}{x^2\left(1+e^{\frac{3}{x}}\right)^2} \qquad \forall x \neq 0.$$

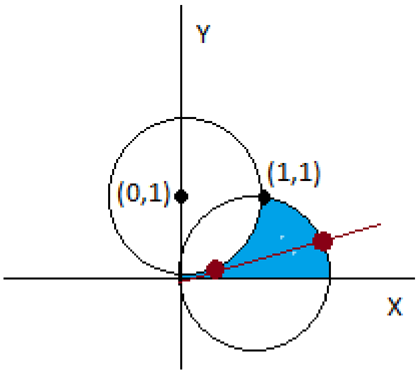
3º. $$\frac{\partial z}{\partial x} = \operatorname{sen}\left(\frac{2x+y}{2x-y}\right)\left(\frac{2(2x-y)-2(2x+y)}{(2x-y)^2}\right) = \frac{-4y}{(2x-y)^2}\operatorname{sen}\left(\frac{2x+y}{2x-y}\right).$$

$$\frac{\partial z}{\partial y} = \text{sen}\left(\frac{2x + y}{2x - y}\right)\left(\frac{(2x - y) + (2x + y)}{(2x - y)^2}\right) = \frac{4x}{(2x - y)^2}\text{sen}\left(\frac{2x + y}{2x - y}\right).$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{-4xy}{(2x - y)^2}\text{sen}\left(\frac{2x + y}{2x - y}\right) + \frac{4xy}{(2x - y)^2}\text{sen}\left(\frac{2x + y}{2x - y}\right) = 0.$$

4º. $x^2 + y^2 - 2y \geq 0 \Rightarrow x^2 + (y - 1)^2 \geq 1.$

$x^2 + y^2 - 2x \leq 0 \Rightarrow (x - 1)^2 + y^2 \leq 1.$




Pasando a coordenadas polares;

$$\iint_D (r\cos\alpha + r\,\text{sen}\alpha)^2 r\,dr\,d\alpha = \int_0^{\frac{\pi}{4}} d\alpha \int_{2\text{sen}\alpha}^{2\cos\alpha} (1 + \text{sen}2\alpha) r^3 dr \qquad (*)$$

$$\left.\frac{r^4}{4}\right|_{2\text{sen}\alpha}^{2\cos\alpha} = \frac{2^4}{4}\left(\cos^4\alpha - \text{sen}^4\alpha\right) = 4\left(\cos^2\alpha - \text{sen}^2\alpha\right)\left(\cos^2\alpha + \text{sen}^2\alpha\right) = 4\cos 2\alpha.$$

$$(*) = 4\int_0^{\frac{\pi}{4}} (1 + \text{sen}2\alpha)\cos 2\alpha\, d\alpha \Rightarrow \begin{cases}(1 + \text{sen}2\alpha) = t \\ 2\cos 2\alpha\, d\alpha = dt\end{cases}$$

$$= 4\left.\frac{(1 + \text{sen}2\alpha)^2}{4}\right|_0^{\frac{\pi}{4}} = (1 + 1)^2 - 1 = 3\ u^3 \text{ (es un volumen)}.$$

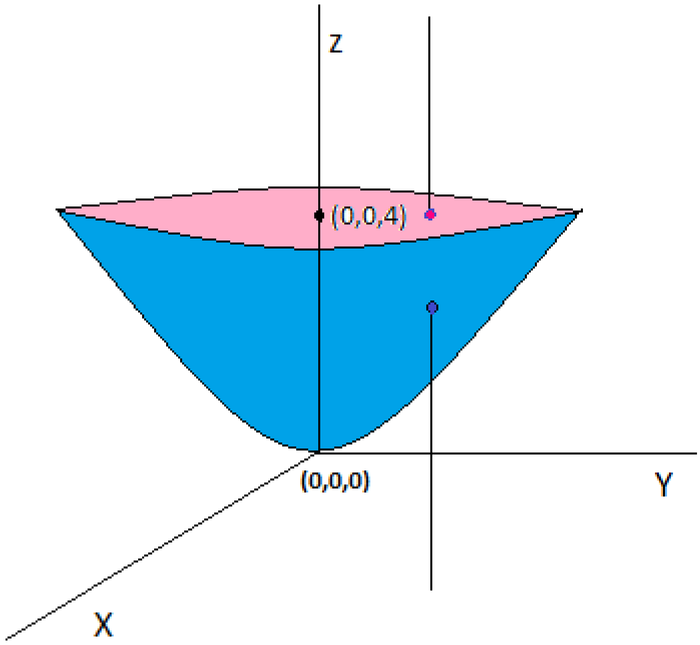
5º. Haciendo el cambio:

$$\left.\begin{matrix} x = 3X \\ y = 2Y \\ z = z \end{matrix}\right\} \frac{\partial(x, y, z)}{\partial(X, Y, z)} = 6 \Rightarrow 6\iiint_W (z^2 + 1)\,dX\,dY\,dz \quad \text{donde W viene}$$

limitado

por:

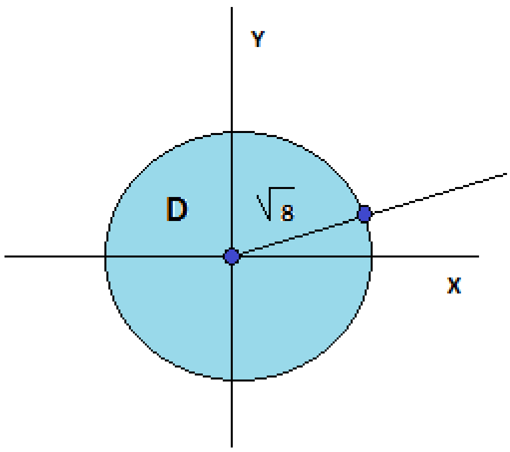
$$\left.\begin{array}{l} X^2 + Y^2 = 2z \\ z = 4 \end{array}\right\}$$




$$6\iiint_W (z^2+1)\,dX\,dY\,dz = \iint_D dXdY \int_{\frac{X^2+Y^2}{2}}^{4} (z^2+1)\,dz.$$

$$\int_{\frac{X^2+Y^2}{2}}^{4} (z^2+1)\,dz = \frac{z^3}{3} + z\Big|_{\frac{X^2+Y^2}{2}}^{4} = \frac{4^3}{3} + 4 - \frac{1}{3}\left(\frac{X^2+Y^2}{2}\right)^3 - \frac{X^2+Y^2}{2}$$

Pasando a coordenadas polares donde D viene dado por:



$$6\int_0^{2\pi} d\alpha \int_0^{\sqrt{8}} \left(\frac{76}{3} - \frac{1}{24}r^6 - \frac{1}{2}r^2\right) r\,dr = 6.2\pi \left(\frac{76}{3.2}r^2 - \frac{1}{24.8}r^8 - \frac{1}{2.4}r^4\right)\Bigg|_0^{\sqrt{8}} = 12\pi.72$$

.

6º. Haciendo el cambio:

$$1 + e^{3x} = t^2 \Rightarrow \begin{cases} 3e^{3x}dx = 2tdt \\ x = \frac{1}{3}\ln(t^2 - 1) \end{cases}$$

$$\int \frac{xe^{3x}}{\sqrt{1 + e^{3x}}} dx = \frac{2}{3.3}\int \frac{t.\ln(t^2 - 1)dt}{\sqrt{t^2}} = \frac{2}{9}\int \ln(t^2 - 1)dt = \text{por partes} =$$

$$= \frac{2}{9}\left(t\ln(t^2 - 1) - \int \frac{2t^2}{t^2 - 1}dt\right).$$

$$2\int \frac{t^2}{t^2 - 1}dt = 2\int \left(1 + \frac{1}{t^2 - 1}\right)dt = 2\left(t + \int \left[\frac{A}{t - 1} + \frac{B}{t + 1}\right]dt\right) =$$

$$2\left(t + \int \left[\frac{1}{2(t - 1)} - \frac{1}{2(t + 1)}\right]dt\right) = 2t + \ln\left(\frac{t - 1}{t + 1}\right).$$

Deshaciendo los cambios:

$$= \frac{2}{9}\left[3x\sqrt{1 + e^{3x}} - 2\sqrt{1 + e^{3x}} - \ln\left(\frac{\sqrt{1 + e^{3x}} - 1}{\sqrt{1 + e^{3x}} + 1}\right)\right] + \text{Cte} =$$

$$= \frac{2}{9}\left[3x\sqrt{1 + e^{3x}} - 2\sqrt{1 + e^{3x}} - \ln\left(\frac{(\sqrt{1 + e^{3x}} - 1)^2}{e^{3x}}\right)\right] + \text{Cte} =$$

$$= \frac{2}{9}\left[3x\left(\sqrt{1 + e^{3x}} - 1\right) - 2\sqrt{1 + e^{3x}} - 2\ln\left(\sqrt{1 + e^{3x}} - 1\right)\right] + \text{Cte}.$$